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Regular admissible wealth processes are necessarily of Black-Scholes type

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Abstract: We show that for a complete market where the stock price uncertainty is driven by a Brownian motion, there exists only one admissible wealth process which is a regular deterministic function of the time and the stock price. In particular, if the stock price is modeled by geometric Brownian motion then the Black-Scholes process is the only regular admissible wealth process.

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1 Introduction

Standard rigorous theories of stock option pricing introduce a hedging portfolio consisting of the underlying risky stock and a riskless bond. The price of the option is then defined as the smallest initial endowment for which there exists an admissible wealth process for the hedging portfolio, that is, for which there exists a self-financing trading strategy for the hedging portfolio in which the resulting wealth process attains the desired value process at the final time. We show that in a complete market with mild smoothness assumptions on the stock's dispersion and the bond's interest rate, current stock option pricing theories have undesirable rigidity; they exclude all but one admissible wealth process which is a "regular" deterministic function of the time and the current stock price, i.e. one which is sufficiently smooth to admit an application of Itô's lemma and which has polynomial growth in the stock price variable. In particular, when the stock price is modeled by geometric Brownian motion, the Black-Scholes process is the only admissible wealth process which is a regular deterministic function of the time and the current stock price.

2 Market Assumptions, European Options and Contingent Claims

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space in which a filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$ and a Brownian motion $W = \{W_t : t \geq 0\}$ are given. We will assume throughout this paper that the Brownian process W is adapted to $\mathcal{F}$ and $\mathcal{F}_0$ contains every $N \in \mathcal{A}$ such that $\mathbb{P}(N) = 0$. Consider a market with two assets which are traded continuously on a time horizon $0 \leq T < \infty$. The first asset, called a *bond*, is riskless with price $\hat{S}(t)$ which evolves according to the equation

$$d\hat{S}(t) = r(t)\hat{S}(t)dt. \tag{2.1}$$

The second asset, called a *stock*, is risky with price S_t modeled by the linear stochastic differential equation

$$dS_t = \mu_t S_t dt + \sigma(t) S_t dW_t. \tag{2.2}$$

The bond's interest rate $r = r(t)$ and the stock's dispersion $\sigma = \sigma(t)$ are assumed Lebesgue measurable and bounded on $[0, T]$, and the stock's mean rate of return process $\{\mu_t, \mathcal{F}_t : 0 \leq t \leq T\}$ is assumed measurable, adapted, and uniformly bounded on $[0, T] \times \Omega$. Furthermore, we assume that the market is *complete*; i.e. there is an $\epsilon > 0$ such that $\sigma(t) \geq \epsilon$ for all $0 \leq t \leq T$.

A *European call option* on the stock is a contract starting at time $t = 0$ which gives the holder the right to buy at a specified time $T > 0$, the *expiration date*, one share of the stock at a specified price q, the *exercise price*. A European call option is a special case of a European contingent claim. A *European contingent claim* (T, f_T, g) is a financial instrument consisting of a payoff rate per unit time $g = \{g_t, \mathcal{F}_t : 0 \leq t \leq T\}$ and a terminal payoff f_T. Here g is a nonnegative, measurable, and adapted process and f_T is a nonnegative, $\mathcal{F}_t$-measurable random variable. In order to ensure square integrability of a martingale in a subsequent result (Theorem 3.1), we will assume that

$$E\left[f_T + \int_0^T g_t dt\right]^2 < \infty.$$

Clearly, a European call option is a European contingent claim with $g \equiv 0$ and $f_T = (S_T - q)^+$.

3 Contingent Claim Valuation and Hedging Strategies

Consider the problem of determining the fair value at time $t = 0$ for a European contingent claim (T, f_T, g). Following [1, 3, 7, 9], the contingent claim is simulated by *a hedging strategy* (π, C) consisting of *a portfolio process* $\pi = \{\pi_t, \mathcal{F}_t : 0 \leq t \leq T\}$ based on the stock and bond and a *consumption process* $C = \{C_t, \mathcal{F}_t : 0 \leq t \leq T\}$.

More precisely, suppose that an investor with an initial endowment $x \geq 0$ invests in the stock and bond. Let $N(t)$ and $\widehat{N}(t)$ denote the number of shares of the stock and the bond, respectively, owned by the investor at time t. The *wealth process* of the investor at time t is then described by $X_0 = x$ and

$$X_t = N(t)S_t + \widehat{N}(t)\hat{S}(t)$$

for $0 < t \leq T$. If trading occurs at discrete times, say t and $t + h$, and C_t denotes the cumulative amount consumed by the investor up to time t, then the increment in investor wealth is given by

$$X_{t+h} - X_t = \widehat{N}(t)\left(\hat{S}(t+h) - \hat{S}(t)\right) + N(t)(S_{t+h} - S_t) - (C_{t+h} - C_t).$$

The analogous continuous-time change in wealth is given by the following expression where Itô's differentials replace discrete differences:

$$dX_t = \widehat{N}(t)d\hat{S}(t) + N(t)dS_t - dC_t.$$

Defining the portfolio process by $\pi_t = N(t)S_t$ and using (2.1) and (2.2), it is easy to see that

$$dX_t = r(t)X_t dt + \left(\mu_t - r(t)\right)\pi_t dt + \sigma(t)\pi_t dW_t - dC_t. \tag{3.1}$$

Assume $\pi = \{\pi_t, \mathcal{F}_t : 0 \leq t \leq T\}$ is measurable, adapted, and satisfies $\int_0^T \pi_s^2 ds < \infty$ a.s. $\mathbb{P}$ for every $0 < T < \infty$. Similarly, assume the consumption process $C = \{C_t, \mathcal{F}_t : 0 \leq t \leq T\}$ is progressively measurable with respect to {Ft}, nonnegative, nondecreasing, bounded, and right-continuous with $C_0 = 0$ a.s. $\mathbb{P}$. The process pair (π, C) is called a *hedging strategy for the contingent claim* (T, f_T, g) provided, a.s. $\mathbb{P}$, $C_t = \int_0^t g_s dt$ for $0 \leq t \leq T$ and $X_t = f_T$. Here X is the wealth process (3.1) associated with the pair (π, C) and the initial endowment $X_0 = x$. Furthermore, the hedging strategy (π, C) is called *admissible* for the initial endowment $x \geq 0$ provided it is self-financing; that is, the associated wealth process X satisfies $X_t \geq 0$ for all $0 \leq t \leq T$, a.s. $\mathbb{P}$. In this case, X is also called *admissible*. The fair value at time $t = 0$ of the contingent claim (T, f_T, g) is defined as the smallest value $x \geq 0$ for which there exists an admissible hedging strategy (π, C) for the contingent claim with initial endowment x.

Under the hypothesis of a complete market, a formula for the fair value of a contingent claim and an expression for the wealth process of the associated unique hedging strategy was obtained in [1].

Theorem 3.1. *Let the contingent claim* (T, f_T, g) *be given. Then under the assumption of a complete market, the fair value of the contingent claim is given by the expectation* $\tilde{E}Q$ *of*

$$Q = e^{-\int_0^T r(u)du} f_T + \int_0^T e^{-\int_0^s r(u)du} g_s ds$$

with respect to the probability measure $\widetilde{\mathbb{P}}$ *whose Radon-Nikodym derivative with respect to* $\mathbb{P}$ *is given by*

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{-\int_0^T \sigma(s)^{-1}\big(\mu_s - r(s)\big)dW_s - \frac{1}{2}\int_0^T \sigma(s)^{-2}\big(\mu_s - r(s)\big)^2 ds\right\}.$$

Furthermore, there exists a unique admissible hedging strategy for the contingent claim with initial endowment $x = \tilde{E}Q$ *and wealth process*

$$X_t = \left[e^{-\int_t^T r(u)du} f_T + \int_t^T e^{-\int_t^s r(u)du} g_s ds \,\middle|\, \mathcal{F}_t\right].$$

4 The Result

Let us now consider the special case of the European contingent claim $(T, f_T, 0)$ where the terminal payoff f_T is a continuous function $\rho: [0,\infty) \to: [0,\infty)$ of the stock price at time $T: f_T = \rho(S_T,)$. Note that the stochastic differential equation (2.2) for the stock price has, together with the initial condition S_0, the unique strong solution

$$S_T = S_0 \exp\left\{\int_0^t \left(\mu_s - \frac{1}{2}\sigma^2(s)\right)ds - \int_0^t \sigma(s)dW_s\right\}.$$

Because the process $\theta_t = \big(\sigma(t)\big)^{-1}\big(\mu_t - r(t)\big)$ is bounded, if we set

$$\widetilde{W}_t = W_t + \int_0^t \theta_s ds \tag{4.1}$$

for $0 \le t < \infty$, then the process $\{\widetilde{W}_t, \mathcal{F}_t : 0 \le t \le T\}$ is a Brownian motion on $\big(\Omega, \mathcal{F}_t, \widetilde{\mathbb{P}}_T\big)$ by the Girsanov theorem. Consequently, for any time $t \ge 0$ we may write

$$S_t = S_0 \exp\left\{\int_0^t \left(r(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)d\widetilde{W}_s\right\}, \tag{4.2}$$

and for times $u > t$,

$$S_u = S_t \exp\left\{\int_t^u \left(r(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_t^u \sigma(s)d\widetilde{W}_s\right\}.$$

Hence we can express the wealth process of Theorem 3.1 as

$$X_t = \tilde{E}\left[e^{-\int_t^T r(u)du}\rho(S_T)\,\middle|\,\mathcal{F}_t\right] = e^{-\int_t^T r(u)du}\tilde{E}\left[\rho\left(S_t \exp\left\{\int_t^T \left(r(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_t^T \sigma(s)d\widetilde{W}_s\right\}\right)\middle|\,\mathcal{F}_t\right] =: Y_T.$$

Taking the expectation of Y_T starting at time t with initial value $y = S_t$, we obtain

$$\tilde{E}^{t,y}[Y_T] = e^{-\int_t^T r(u)du}\tilde{E}^{t,y}\left[\rho\left(y\exp\left\{\int_t^T \left(r(s)-\frac{1}{2}\sigma^2(s)\right)ds + \int_t^T \sigma(s)d\widetilde{W}_s\right\}\right)\right] \tag{4.3}$$

$$=: G(t,y). \tag{4.4}$$

Observe that we can write the value of the valuation process for the contingent claim $(T, f_T, 0)$ at each time $t \in [0,T]$ as a deterministic function, dependent only on the time t and the current price $y = S_t$ of the stock.

We can identify this function $G = G(t,x)$ as a solution to a partial differential equation by appealing to a stochastic representation theorem in the spirit of Feynman [4] and Kac [6]. Consider the stochastic integral equation

$$S_s^{t,x} = x + \int_t^s \beta\left(u, S_u^{t,x}\right)du + \int_t^s \alpha\left(u, S_u^{t,x}\right)dW_u \tag{4.5}$$

for $t \le s < \infty$ where $\gamma = \gamma(t,x), \beta = \beta(t,x)$, and $\alpha = \alpha(t,x)$ are continuous on $[0,\infty) \times \mathbb{R}$ and the linear growth condition $|\beta(t,x)|^2 + |\alpha(t,x)|^2 \le K^2(1+x^2)$ is satisfied for some positive constant K and all $0 \le t < T$. For every pair (x,t), let the equation (4.5) have a weak solution which is unique in the sense of probability.

Theorem 4.1. *([8], p. 366) Let $v = v(t,x)$ be continuous on $[0,T] \times \mathbb{R}$, of class $C^{1,2}([0,T) \times \mathbb{R})$, and satisfy the Cauchy final value problem:*

$$-\frac{\partial v}{\partial t} + \gamma(t,x)v = \frac{1}{2}\alpha^2(t,x)\frac{\partial^2 v}{\partial x^2} + \beta(t,x)\frac{\partial v}{\partial x} \tag{4.6}$$

in $[0,T) \times \mathbb{R}$,

$$v(T,x) = f(x)$$

if $x \in \mathbb{R}$, and the polynomial growth condition

$$\max\{|v(t,x)|: 0 \le t \le T\} \le M(1+|x|^{2\mu})$$

for some constants $M > 0$ and $\mu \ge 1$ and all $x \in \mathbb{R}$. Then v admits the stochastic representation

$$v(t,x) = E^{t,x}\left[f\left(S_T^{t,x}\right)\exp\left\{-\int_t^T r\left(s, S_s^{t,x}\right)ds\right\}\right] \tag{4.7}$$

on $[0,T] \times \mathbb{R}$.

Comparing (4.3) with (4.7) and taking $\gamma(t,x) = r(t), \beta(t,x)) = r(t)x$, and $\alpha(t,x) = \alpha(t)x$ in (4.6), we are led to consider the Cauchy final value problem:

$$-\frac{\partial G}{\partial t} + r(t)G = \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 G}{\partial x^2} + r(t)x\frac{\partial G}{\partial x} \tag{4.8}$$

in $[0,T) \times (0,\infty)$,

$$G(T,x) = \rho(x) \tag{4.9}$$

if $x \in (0,\infty)$, and

$$\max\{|G(t,x)|: 0 \le t \le T\} \le M(1+|x|^{2\mu}) \tag{4.10}$$

for some constants $M > 0$ and $\mu \ge 1$ and all $x \in (0,\infty)$. The transformations $y = \ln(x), \tau = T - t$, and $u(\tau,y) = G(t,x)$ convert (4.8), (4.9), and (4.10) into an equivalent Cauchy initial value problem:

$$\frac{\sigma^2(t)}{2}\frac{\partial^2 u}{\partial y^2} + \left(r(t) - \frac{\sigma^2(t)}{2}\right)\frac{\partial u}{\partial y} - r(t)u - \frac{\partial u}{\partial \tau} = 0 \tag{4.11}$$

in $(0,T] \times \mathbb{R}$,

$$u(0,y) = \rho(e^y) \tag{4.12}$$

if $y \in \mathbb{R}$, and

$$\max\{|u(\tau,y)| : 0 \le \tau \le T\} \le M(1 + e^{2\mu y}) \tag{4.13}$$

for all $y \in \mathbb{R}$. Suppose the bond's interest rate $r = r(t)$ and the stock's dispersion $\sigma = \sigma(t)$ are uniformly Hölder continuous on $[0,T]$ for some exponent $\alpha \in (0,1]$, that $\sigma(t) \ge \epsilon > 0$ for all $t \in [0,T]$, and $\rho = \rho(x)$ is a continuous function on $[0,\infty)$ satisfying the growth condition

$$|\rho(x)| \le L \exp(kx^2) \tag{4.14}$$

for some constants $L \ge 0$ and $k > 0$ and all $x \in [0,\infty)$. Standard existence and uniqueness results for parabolic partial differential equations (cf. [5] Chapter 1, Theorems 12 and 16) then guarantee that there is one and only one function $u = u(\tau,y)$ in $C([0,T] \times \mathbb{R})$ which satisfies (4.11), (4.12), and (4.13). Thus (4.8), (4.9), and (4.10) have a unique solution $G = G(t,x)$ and consequently, all the hypotheses of Theorem 4.1 are fulfilled. We conclude that the evaluation process can be expressed as $X_t = G(t,S_t)$ where $G = G(t,x)$ is the unique solution of the Cauchy final value problem (4.8), (4.9), and (4.10). Therefore, the following result (cf. [10]) shows that, under mild smoothness conditions on the stock's dispersion and the bond's interest rate, we can bypass Theorem 3.1.

Theorem 4.2. *Let r and σ be uniformly Hölder continuous functions on $[0,T]$ for some exponent $\alpha \in (0,1]$, let $\sigma(t) \ge \epsilon > 0$ for all $t \in [0,T]$, let ρ be a continuous function on $[0,\infty)$ satisfying the growth condition (4.14), and let X_t denote the wealth process of any admissible hedging strategy $(\pi,0)$ for the contingent claim $(T,\rho(S_t),0)$. If $X_t = G(t,S_t)$, a deterministic function of time and the current stock price, where $G = G(t,x)$ is continuous on $[0,T] \times \mathbb{R}$, is of class $C^{1,2}([0,T] \times \mathbb{R})$, and satisfies the polynomial growth condition (4.10), then G is the unique solution of the Cauchy final value problem (4.8), (4.9), and (4.10).*

Proof. From the assumptions, we know directly that equations (4.9) and (4.10) must hold for G. It remains to show that equation (4.8) holds as well. From equation (4.2) we obtain

$$S_t = e^{\int_0^t r(s)ds} M_t, \tag{4.15}$$

$$M_t := S_0 \exp\left\{\int_0^t \sigma(s) d\widetilde{W}_s - \frac{1}{2}\int_0^t \sigma^2(s) ds\right\}, \tag{4.16}$$

where M_t is a martingale since $\sigma = \sigma(t)$ is bounded, and hence we obtain $dM_t = \sigma(t) M_t d\widetilde{W}_t$. Defining $\Gamma(t,x) := G\left(t, xe^{\int_0^t r(s)ds}\right)$ we have

$$X_t = G(t,S_t) = \Gamma(t,M_t) \tag{4.17}$$

and Itô's lemma implies

$$\begin{aligned} dX_t = d\Gamma(t,M_t) &= \frac{\partial \Gamma}{\partial t}(t,M_t)dt + \sigma(t)M_t \frac{\partial \Gamma}{\partial x}(t,M_t)d\widetilde{W}_t + \frac{1}{2}\sigma^2(t)M_t^2 \frac{\partial^2 \Gamma}{\partial x^2}(t,M_t)dt \\ &= \left(\frac{\partial \Gamma}{\partial t}(t,M_t) + \frac{1}{2}\sigma^2(t)M_t^2 \frac{\partial^2 \Gamma}{\partial x^2}(t,M_t)\right)dt + \sigma(t)M_t \frac{\partial \Gamma}{\partial x}(t,M_t)d\widetilde{W}_t. \end{aligned}$$

Together with

$$\frac{\partial \Gamma}{\partial t}(t,M_t) = \frac{\partial \mathrm{G}}{\partial t}(t,S_t) + r(t)S_t \frac{\partial \mathrm{G}}{\partial x}(t,S_t),$$

$$\frac{\partial \Gamma}{\partial x}(t,M_t) = e^{\int_0^t r(s)ds} \frac{\partial \mathrm{G}}{\partial x}(t,S_t),$$

and

$$\frac{\partial^2\Gamma}{\partial x^2}(t,M_t) = e^{2\int_0^t r(s)ds}\frac{\partial^2 \mathrm{G}}{\partial x^2}(t,S_t),$$

this yields

$$dX_t = \left(\frac{\partial \mathrm{G}}{\partial t}(t,S_t) + r(t)S_t\frac{\partial \mathrm{G}}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2(t)S_t^2\frac{\partial^2 \mathrm{G}}{\partial x^2}(t,S_t)\right)dt + \sigma(t)S_t\frac{\partial \mathrm{G}}{\partial x}(t,S_t)d\widetilde{W}_t. \tag{4.18}$$

But from equations (3.1) and (4.1) we have $dX_t = r(t)G(t,S_t)dt + \sigma(t)S_tN_td\widetilde{W}_t$.

Equating coefficients with (4.18) gives

$$\sigma(t)S_tN_t = \sigma(t)S_t\frac{\partial \mathrm{G}}{\partial x}(t,S_t)$$

so that

$$N_t = \frac{\partial \mathrm{G}}{\partial x}(t,S_t)$$

and

$$r(t)G(t,S_t) = \frac{\partial \mathrm{G}}{\partial x}(t,S_t) + r(t)S_t\frac{\partial \mathrm{G}}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2(t)S_t^2\frac{\partial^2 \mathrm{G}}{\partial x^2}(t,S_t).$$

Since $\mathbb{P}(x-\epsilon \le S_t \le x+\epsilon) > 0$ for every $x \ge 0$ and for every $\epsilon > 0$, the function G must satisfy equation (4.8).

5 Summary Observations on the Results

Assume that the market is complete, $\rho: [0,\infty) \to [0,\infty)$ is continuous, and the stock price is given by the linear stochastic differential equation (2.2). Bensoussan's result - Theorem 3.1 - shows that among all admissible hedging strategies $(\pi, 0)$ for the European contingent claim $(T, \rho(S_t), 0)$ there exists a unique one with smallest initial endowment. Moreover, the discussion in Section 4 shows that the value of the admissible wealth process $X = \{X_t, \mathcal{F}_t; 0 \le t \le T\}$ at each time $t \in [0,T]$ associated with this minimal admissible hedging strategy is a deterministic function of the time t and the current price St of the stock: $X_t = G(t,S_t)$.

On the other hand, Theorem 4.2 shows that, under mild smoothness conditions on the stock's dispersion and the bond's interest rate and mild smoothness and growth conditions on the terminal payoff, there exists only one admissible wealth process $X = \{X_t, \mathcal{F}_t; 0 \le t \le T\}$ for the European contingent claim $(T, \rho(S_t), 0)$ of the form $X_t = G(t,S_t)$ in which G is regular. In fact, using a stochastic representation in the spirit of Feynman and Kac - Theorem 4.1 - G is identified as the unique solution to the Cauchy final value problem (4.8), (4.9), and (4.10). In the special case when the bond's interest rate is constant, $r(t) = r > 0$, the stock's mean rate of return and dispersion are constant, $\mu_t = \mu > 0$ and $\sigma(t) = \sigma > 0$, and the terminal payoff models a European stock option with exercise price $q, \rho(S_t) = (S_t - q)^+$, then (2.2) becomes geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

and the solution of (4.8), (4.9), and (4.10) is given by the classical Black- Scholes formula [2]:

$$G(t,x) = \begin{cases} x\Phi\big(\lambda_+(T-t,x)\big) - qe^{-r(T-t)}\Phi\big(\lambda_-(T-t,x)\big) & if\ 0 \le t < T, 0 \le x < \infty, \\ (x,q)^+ & if\ t = T, 0 \le x < \infty, \end{cases}$$

where

$$\lambda_\pm(t,x) = \frac{1}{\sigma\sqrt{t}}\left(\ln\left(\frac{x}{q}\right) + t\left(r \pm \frac{\sigma^2}{2}\right)\right)$$

and

$$\Phi(\mathrm{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz.$$

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