New Trends in Mathematical Sciences
http://www.ntmsci.com

A study on the nonlinear fractional generalized reaction duffing model

Ozkan Guner[1] *and Murat Atik*[2]

[1]Cankiri Karatekin University, Department of International Trade,Cankiri-Turkey
[2]Cankiri Karatekin University, Department of Banking and Finance,Cankiri-Turkey

Received: 5 May 2015, Revised: 2 June 2015, Accepted: 17 June 2015
Published online: 5 December 2015

Abstract: In this paper, an efficient method namely $\left(\frac{G'}{G}\right)$-expansion method for solving the fractional generalized reaction duffing model is considered. The fractional derivative is described in the Jumarie's modified Riemann– Liouville sense. As a result, we obtain the hyperbolic and periodic function solutions of this equation. The results obtained by this method have been compared with the other solutions show that proposed method is accuracy and convenience for solving nonlinear fractional differential equations.

Keywords: Modified Riemann-Liouville derivative, fractional generalized reaction duffing model, the $\left(\frac{G'}{G}\right)$-expansion method.

1 Introduction

Nonlinear fractional differential equations have shown to be adequate models for many important phenomena in physics, engineering, electromagnetic, mathematics and material science.[1,2,3]. The exact and approximate solutions for nonlinear fractional differential equations may guide peoples to better understand relevant phenomena and further apply them in practical scientific research. In recent years, the analytical solution of fractional differential equations have been devoted a lot of attention of specialists and scholars's interest. Fistly, a promising analytic approach called the fractional sub-equation method [4,5,6] has been successfully applied to solve many kinds of nonlinear fractional differential equations. Then several mathematical methods such as the fractional exp-function method [7,8,9,10], the fractional first integral method [11,12], the fractioanal modified trial equation method [13], the fractional $\left(\frac{G'}{G}\right)$-expansion method [14, 15,16,17], the fractional fractional functional variable method [18,19] and the fractional simplest equation method [20] have been developed to obtain exact analytic solutions. We notice that the method relies on the homogeneous balance principle [21] and the symbolic computation.

There are several definitions of a fractional derivative of order α. Most commonly used definitions are the modified Riemann–Liouville and Caputo [22,23]. We firstly give some properties and definitions of the modified Riemann–Liouville derivative which are used further in this paper. Jumarie proposed a modified Riemann–Liouville derivative. Assume that $f : R \to R,\ t \to f(t)$ denote a continuous (but not necessarily differentiable) function. The

* Corresponding author e-mail: ozkanguner@karatekin.edu.tr

modified Riemann–Liouville derivative of order α is defined by the expression [24]

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\xi)^{-\alpha-1}[f(\xi)-f(0)]d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha \leq n+1, \quad n \geq 1, \end{cases} \tag{1}$$

and its some useful formulas of them are

$$D_t^{\alpha} x^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha} \quad ,\gamma > 0, \tag{2}$$

$$D_t^{\alpha}(cf(t)) = cD_t^{\alpha} f(t), \quad c = \text{constant} \tag{3}$$

$$D_t^{\alpha}\{af(t)+bg(t)\} = aD_t^{\alpha} f(t) + bD_t^{\alpha} g(t), \tag{4}$$

where a and b constant.

$$D_t^{\alpha} c = 0, \quad c = \text{constant} \tag{5}$$

which are direct consequences of the equality $d^{\alpha}x(t) = \Gamma(1+\alpha)dx(t)$.

The organization of this paper is as follows. In section 2, the description of the $\left(\frac{G'}{G}\right)$-expansion method is given for solving fractional partial differential equations. Then in section 3 this method is applied to establish exact solutions for the fractional generalized reaction duffing model. Some conclusions are given in last section.

2 Algorithm of the proposed method for FDEs

In the following, we give the main steps of the $\left(\frac{G'}{G}\right)$-expansion method for solving fractional differential equations. We consider the following nonlinear FDE of the type

$$F(u, D_t^{\alpha} u, D_x^{\beta} u, D_t^{\alpha} D_t^{\alpha} u, D_t^{\alpha} D_x^{\beta} u, D_x^{\beta} D_x^{\beta} u, \ldots) = 0, \quad 0 < \alpha, \beta < 1 \tag{6}$$

where u is an unknown function, and P is a polynomial of u and its partial fractional derivatives.

Step 1: Li and He [25] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODEs). The traveling wave variable

$$u(x,t) = U(\xi), \tag{7}$$

$$\xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{8}$$

where c and k are non zero arbitrary constants. By using the chain rule

$$D_t^{\alpha}u = \sigma_t^{'}\frac{dU}{d\xi}D_t^{\alpha}\xi$$
$$D_x^{\alpha}u = \sigma_x^{'}\frac{dU}{d\xi}D_x^{\alpha}\xi \tag{9}$$

where $\sigma_t^{'}$ and $\sigma_x^{'}$ are called the sigma indexes see [26,27], without loss of generality we can take $\sigma_t^{'} = \sigma_x^{'} = l$, where l is a constant.

Substituting (8) with (2) and (9) into (6), we can rewrite Eq. (6) in the following nonlinear ODE;

$$Q(U,U^{'},U'',U''',.....) = 0, \tag{10}$$

where the prime denotes the derivation with respect to ξ.

Step 2: Suppose that the solution of equation (10) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \quad a_m \neq 0, \tag{11}$$

where a_i $(i = 0,1,2,.....,m)$ are constants, while $G(\xi)$ satisfies the second ordinary differential equation in the form

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{12}$$

and λ, μ and a_i $(i = 0,1,2,.....,m)$ are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (10). By the generalized solutions of Eq. (12) we have

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) - \frac{\lambda}{2}, & \lambda^2-4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right) - \frac{\lambda}{2}, & \lambda^2-4\mu < 0, \\ -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}, & \lambda^2-4\mu = 0, \end{cases} \tag{13}$$

where C_1, C_2 are arbitrary constants.

Step 3: Substituting equation (11) into equation (10) and using equation (13) collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together. Then equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for a_i $(i = 0,1,2,.....,m)$, λ, μ, k_1, k_2, k_3, ...and c.

Step 4: Solving the equations system in Step 3, and using equation then substituting a_i $(i = 0,1,2,.....,m)$, λ, μ, k_1, k_2, k_3,...,c and the general solutions of equation (13) into equation (11), we can get a variety of exact solutions of equation (6) [28,29,30].

3 Applications

In this section, we consider the exact solutions of fractional generalized reaction duffing model can be given as follows [31,32]

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+p\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+qu(x,t)+ru^2(x,t)+su^3(x,t)=0, \quad t>0,\ 0<\alpha\leq 1, \tag{14}$$

where $p,\ q,\ r$ and s are all constants. If we take $r=0$ Eq. (14) reduces

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+p\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+qu(x,t)+su^3(x,t)=0, \quad t>0,\ 0<\alpha\leq 1, \tag{15}$$

nonlinear wave equation. Substituting (8) with (2) and (9) into (15), Eq. (15) reduced into an ODE

$$c^2U^{''}+pk^2U''+qU+sU^3=0, \tag{16}$$

where $U'=\frac{dU}{d\xi}$.

We can determine value of m by balancing $U^{''}$ and U^3 in Eq.(16). We find

$$\begin{aligned} m+2&=3m, \\ m&=1. \end{aligned} \tag{17}$$

We can suppose that the solutions of Eq. (16) is of the form

$$U(\xi)=a_0+a_1\left(\tfrac{G'}{G}\right), \quad a_1\neq 0. \tag{18}$$

By using Eq. (18) and (13) we have

$$U''(\xi)=2a_1\left(\frac{G'}{G}\right)^3+3a_1\lambda\left(\frac{G'}{G}\right)^2+(2a_1\mu+a_1\lambda^2)\left(\frac{G'}{G}\right)+a_1\lambda\mu, \tag{19}$$

and

$$U^3(\xi)=a_1^3\left(\frac{G'}{G}\right)^3+3a_0a_1^2\left(\frac{G'}{G}\right)^2+3a_0^2a_1\left(\frac{G'}{G}\right)+a_0^3. \tag{20}$$

Substituting Eq. (18)-(20) into Eq. (16), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ $(i=0,\ldots,3)$ and set it to zero we obtain the system

$$\begin{aligned} &c^2a_1\lambda\mu+pk^2a_1\lambda\mu+qa_0+sa_0^3=0, \\ &(\lambda^2+2\mu)a_1c^2+p(\lambda^2+2\mu)a_1k^2+qa_1+3sa_0^2a_1=0, \\ &3c^2a_1\lambda+3pk^2a_1\lambda+3sa_0a_1^2=0, \\ &2c^2a_1+2pk^2a_1+sa_1^3=0. \end{aligned} \tag{21}$$

Solving this system by using Maple gives

$$\begin{array}{cc} a_0 = \pm\lambda\sqrt{\frac{q}{(4\mu-\lambda^2)s}} & a_1 = \pm 2\sqrt{\frac{q}{(4\mu-\lambda^2)s}} \\ k = k, & c = \pm\sqrt{-pk^2 + \frac{2q}{\lambda^2-4\mu}}. \end{array} \tag{22}$$

where λ and μ are arbitrary constants. By using Eq. (18), expression (22) can be written as

$$U(\xi) = \pm\lambda\sqrt{\frac{q}{(4\mu-\lambda^2)s}} \pm 2\sqrt{\frac{q}{(4\mu-\lambda^2)s}}\left(\frac{G'}{G}\right). \tag{23}$$

Substituting Eq. (13) into Eq. (23) we have two types of exact solutions of this equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$U_{1,2}(\xi) = \pm\sqrt{-\frac{q}{s}}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right), \tag{24}$$

where $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} \mp \sqrt{-pk^2 + \frac{2q}{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

When $\lambda^2 - 4\mu < 0$,

$$U_{3,4}(\xi) = \pm\sqrt{\frac{q}{s}}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right), \tag{25}$$

where $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} \mp \sqrt{-pk^2 + \frac{2q}{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

In particular, if $C_1 \neq 0,\ C_2 = 0,\ \lambda > 0,\ \mu = 0$, then $U_{1,2}$ and $U_{3,4}$ become

$$u_{1,2}(x,t) = \pm\sqrt{-\frac{q}{s}}\tanh\left(\frac{\lambda kx^\alpha}{2\Gamma(1+\alpha)} \mp \frac{\sqrt{2q - p\lambda^2k^2}t^\alpha}{2\Gamma(1+\alpha)}\right). \tag{26}$$

Remark 1. Eq. (14) reductions many well-known nonlinear fractional wave equations such as:

(i). $p = -1,\ q = -m^2,\ r = 0$ and $s = g^2$, Eq. (14) reduces to the fractional Landau-Ginzburg-Higgs equation.

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} - m^2u(x,t) + g^2u^3(x,t) = 0, \quad t > 0,\ 0 < \alpha \leq 1, \tag{27}$$

and its exact solutions

$$u_{3,4}(x,t) = \pm\frac{m}{g}\tanh\left(\frac{\lambda kx^\alpha}{2\Gamma(1+\alpha)} \mp \frac{\sqrt{2q + \lambda^2k^2}t^\alpha}{2\Gamma(1+\alpha)}\right). \tag{28}$$

(ii). $p = -1,\ q = -a,\ r = 0$ and $s = -b$, Eq. (14) reduces to the classical fractional Klein-Gordon equation.

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} - au(x,t) - bu^3(x,t) = 0, \quad t > 0,\ 0 < \alpha \leq 1, \tag{29}$$

and its exact solutions

$$u_{5,6}(x,t) = \pm\sqrt{-\frac{a}{b}}\tanh\left(\frac{\lambda kx^\alpha}{2\Gamma(1+\alpha)} \mp \frac{\sqrt{-2a + \lambda^2k^2}t^\alpha}{2\Gamma(1+\alpha)}\right). \tag{30}$$

(iii). $p=-1,\ q=1,\ r=0$ and $s=-1$, Eq. (14) reduces to the ϕ^4 equation.

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+u(x,t)-u^3(x,t)=0,\quad t>0,\ 0<\alpha\leq 1, \tag{31}$$

and its exact solutions

$$u_{7,8}(x,t)=\pm\tanh\left(\frac{\lambda kx^{\alpha}}{2\Gamma(1+\alpha)}\mp\frac{\sqrt{2+\lambda^2k^2}t^{\alpha}}{2\Gamma(1+\alpha)}\right). \tag{32}$$

(iv). $p=-1,\ q=1,\ r=0$ and $s=-\frac{1}{6}$, Eq. (14) reduces to the Sine-Gordon equation.

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+u(x,t)-\frac{1}{6}u^3(x,t)=0,\quad t>0,\ 0<\alpha\leq 1, \tag{33}$$

and its exact solutions

$$u_{9,10}(x,t)=\pm\sqrt{6}\tanh\left(\frac{\lambda kx^{\alpha}}{2\Gamma(1+\alpha)}\mp\frac{\sqrt{2+\lambda^2k^2}t^{\alpha}}{2\Gamma(1+\alpha)}\right). \tag{34}$$

(v). $p=0,\ q=a,\ r=0$ and $s=b$, Eq. (14) reduces to the Duffing equation.

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+au(x,t)+bu^3(x,t)=0,\quad t>0,\ 0<\alpha\leq 1, \tag{35}$$

also its exact solutions

$$u_{11,12}(x,t)=\pm\sqrt{-\frac{a}{b}}\tanh\left(\frac{\lambda kx^{\alpha}}{2\Gamma(1+\alpha)}\mp\frac{\sqrt{2a}t^{\alpha}}{2\Gamma(1+\alpha)}\right). \tag{36}$$

Remark 2. Comparing our results to Mirzazadeh's and Bekir's results [32,33], it can be seen that the results are same when we choose some proper values. In [31], our results different than Jafari's result.

4 Conclusion

In this paper, the fractional complex transform can easily convert a fractional differential equation into its equivalent ordinary differential equation. Then, the $\left(\frac{G'}{G}\right)$-expansion method has been successfully employed to obtain the exact solution of the fractional generalized reaction duffing model. The obtained solutions demonstrate the reliability of the algorithm and its wider applicability to nonlinear fractional partial differential equations. These solutions include the generalized hyperbolic function solutions, generalized trigonometric function solutions, and rational function solutions, which may be very useful to understand the nonlinear FDEs. From the results we seen that the proposed method is a very effective and powerful technique in determining exact solutions of time fractional nonlinear partial differential equations.

References

[1] Miller, K.S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, (1993).

[2] Podlubny, I., Fractional Differential Equations, Academic Press, California, (1999).

[3] Kilbas, A.A., Srivastava, H.M., Trujillo J J Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, (2006).

[4] Zhang, S., Zhang, H-Q., Fractional sub-equation method and its applications to nonlinear fractional PDEs, Physics Letters A, 375 (2011) 1069–1073.

[5] Liu, Y., Yan, L., Solutions of Fractional Konopelchenko-Dubrovsky and Nizhnik-Novikov-Veselov Equations Using a Generalized Fractional Subequation Method, Abstract and Applied Analysis, 2013 (2013) 839613.

[6] Alzaidy, J. F., Fractional Sub-Equation Method and its Applications to the Space–Time Fractional Differential Equations in Mathematical Physics, British Journal of Mathematics & Computer Science, 3 (2013) 153-163.

[7] Zhang, S., Zong Q-A., Liu, D., Gao, Q., A Generalized Exp-Function Method for Fractional Riccati Differential Equations, Communications in Fractional Calculus, 1 (2010) 48-51.

[8] Guner, O., Cevikel, A.C., A Procedure to Construct Exact Solutions of Nonlinear Fractional Differential Equations, The Scientific World Journal, 2014 (2014) 489495.

[9] Zheng, B., Exp-Function Method for Solving Fractional Partial Differential Equations, The ScientificWorld Journal, 2013, 465723 (2013).

[10] Bekir, A., Guner, O., Analytical Approach for the Space-time Nonlinear Partial Differential Fractional Equation, International Journal of Nonlinear Sciences and Numerical Simulation, 15 7-8 (2014) 463–470.

[11] Lu, B., The first integral method for some time fractional differential equations, J. Math. Anal. Appl., 395 (2012) 684–693.

[12] Bekir, A., Guner, O., Unsal, O., The First Integral Method for Exact Solutions of Nonlinear Fractional Differential Equations, Journal of Computational and Nonlinear Dynamics, 10 021020-5 (2015) 463–470.

[13] Bulut, H., Baskonus, H M., Pandir, Y., The Modified Trial Equation Method for Fractional Wave Equation and Time Fractional Generalized Burgers Equation, Abstract and Applied Analysis, 2013 636802 2013.

[14] Zheng, B., (G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics, Commun. Theor. Phys., 58 (2012) 623–630.

[15] Guner, O., Eser, D., Exact Solutions of the Space Time Fractional Symmetric Regularized Long Wave Equation Using Different Methods, Advances in Mathematical Physics, 2014 (2014) 456804.

[16] Gepreel, K. A., Omran, S., Exact solutions for nonlinear partial fractional differential equations, Chin. Phys. B, 21 (2012) 110204.

[17] Bekir, A., Guner, O., Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method, Chin. Phys. B, 22, 11 (2013) 11020.

[18] Liu, W., Chen, K., The functional variable method for finding exact solutions of some nonlinear time-fractional differential equations, Pramana – J. Phys., 81 (2013) 3.

[19] Bekir, A., Guner, O., Aksoy, E., Pandir, Y., Functional variable method for the nonlinear fractional differential equations, AIP Conf. Proc., 1648 (2015) 730001.

[20] Taghizadeh, N., Mirzazadeh, M., Rahimian, M., Akbari, M., Application of the simplest equation method to some time-fractional partial differential equations, Ain Shams Engineering Journal, 4 4 (2013) 897–902.

[21] Wang, M.L., Solitary wave solutions for variant Boussinesq equations, Physics Letters A, 199 (1995) 169–172.

[22] Jumarie, G., Modified Riemann–Liouville derivative and fractional Taylor series of nondifferentiable functions further results, Comput. Math. Appl., 51 (2006) 1367–1376.

[23] Caputo, M., Linear models of dissipation whose Q is almost frequency independent II, Geophys. J. Royal Astronom. Soc, 13, (1967) 529-539.

[24] Jumarie, G., Table of some basic fractional calculus formulae derived from a modified Riemann-Liouvillie derivative for nondifferentiable functions, Appl. Maths. Lett., 22 (2009) 378-385.

[25] He, J H., Elegan, S.K., Li, Z.B.,Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus, Physics Letters A, 376 (2012) 257–259.

[26] Saad, M., Elagan, S.K., Hamed, Y.S., Sayed, M., Using a complex transformation to get an exact solutions for fractional generalized coupled MKDV and KDV equations, Int. Journal of Basic & Applied Sciences, 13 01 (2013) 23–25.

[27] Elghareb, T., Elagan, S.K., Hamed, Y.S., Sayed, M., An Exact Solutions for the Generalized Fractional Kolmogrove-Petrovskii Piskunov Equation by Using the Generalized (G'/G)-expansion Method, Int. Journal of Basic & Applied Sciences, 13 01 (2013) 19–22.

[28] Wang, M., Li, X., Zhang, J., The (G'/G)-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics Phys. Lett., A 372 (2008) 417-423.

[29] Bekir, A., Application of the (G'/G)-expansion method for nonlinear evolution equations, Physics Letters, A 372 (2008) 3400-3406.

[30] Zayed, E.M.E., Gepreel, K.A., The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics J. Math. Phys., 50 (2009) 013502.

[31] Jafari, H., Tajadodi, H., Baleanu, D., Al-Zahrani, A.A., Alhamed, Y.A., Zahid, A.H., Fractional sub-equation method for the fractional generalized reaction Duffing model and nonlinear fractional Sharma-Tasso-Olver equation, Cent. Eur. J. Phys., 11 10 (2013) 1482-1486.

[32] Eslami, M., Vajargah, B.F., Mirzazadeh, M.,.Biswas, A., Application of first integral method to fractional partial differential equations, Indian J Phys., 88 2 (2014)177–184.

[33] Guner, O., Bekir, A., Exact solutions of some fractional differential equations arising in mathematical biology, International Journal of Biomathematics, 8 1 (2015) 1550003.