New Trends in Mathematical Sciences
http://www.ntmsci.com

A new algorithm for the numerical solution of the first order nonlinear differential equations with the mixed non-linear conditions by using Bernstein polynomials

Huriye Gurler and Salih Yalcinbas

Department of Mathematics, Celal Bayar University, Manisa, Turkey

Received: 4 March 2015, Revised: 23 March 2015, Accepted: 9 April 2015
Published online: 4 December 2015

Abstract: In this study, an approximate method based on Bernstein polynomials has been presented to obtain the solution first order nonlinear ordinary differential equations with the mixed non-linear conditions. The method by means of Bernstein collocation points, transforms the differential equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Bernstein coefficients in [1, 2]. The method can be used for solving Riccati equation as well. The numerical results show the applicability of the method for this type of equations. Comparisons are made between the obtained solution and the exact solution.

Keywords: Nonlinear ordinary differential equations, Riccati equation, Bernstein polynomials, Collocation points.

1 Introduction

Problems in many areas of science like chemical reactions, resistor-capacitor-inductance, spring mass systrems bending of beams, pendulums, the motion of a rotating mass around another body, etc. can be solved by nonlinear ordinary differential equations. Same kind of equations can also be used in ecology, economics, biology, astrophysics and engineering. That is why these methods are important to Engineers and scientists[3-8].

We consider the first order nonlinear ordinary differential equation of the form

$$P(x)y(x)+Q(x)y'(x)+R(x)y^2(x)+S(x)y(x)y'(x)+T(x)\left(y'(x)\right)^2=g(x) \tag{1}$$

under the mixed non-linear conditions

$$\alpha y(a)+\beta y(b)+\gamma(y(a))^2+\tau(y(b))^2+\xi y(a)y(b)=\lambda \tag{2}$$

where $P(x), Q(x), R(x), S(x), T(x)$ and $g(x)$ are the functions defined on ; the real coefficients $\alpha, \beta, \gamma, \tau, \xi$ and λ are appropriate constants. Our aim is to seek the approximate solution in the form,

* Corresponding author e-mail: salih.yalcinbas@cbu.edu.tr

$$y(x) \cong \sum_{n=0}^{N} y_n B_{n,N}(x), \tag{3}$$

$$B_{n,N}(x) = \sum_{k=0}^{N-n} \frac{(-1)^k}{R^{n+k}} \binom{N}{n} \binom{N-n}{k} x^{n+k}, \ (n = 0, 1, ..., N), \ 0 \le x \le R,$$

where $y_n, \ (n = 0, 1, ..., N)$ are the coefficients to be determined and $B_{n,N}(x)$ is the Bernstein polynomial of degree N.

2 Fundamental matrix relations

Let us consider the nonlinear differential equation (1) and find the matrix forms of each term in these equations. Firstly, we consider the solution $y(x)$ defined by a truncated series (3) and then we can convert it to the matrix form,

$$y(x) = \mathbf{B}(x)\mathbf{Y} \tag{4}$$

where

$$\mathbf{B}(x) = \begin{bmatrix} B_{0,N}(x) & B_{1,N}(x) & \cdots & B_{N,N}(x) \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} y_0 & y_1 & \cdots & y_N \end{bmatrix}^T.$$

If we differentiate equation (4) with respect to x, we obtain

$$y'(x) = \mathbf{B}'(x)\mathbf{Y} = \mathbf{B}(x)\mathbf{H}\mathbf{Y} \tag{5}$$

so that

$$\mathbf{H} = (\mathbf{M}^T)^{-1}\Pi\mathbf{M}^T$$

$$\mathbf{M} = \begin{bmatrix} m_{00} & m_{01} & \cdots & m_{0N} \\ m_{10} & m_{11} & \cdots & m_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N0} & m_{N1} & \cdots & m_{NN} \end{bmatrix}, \ m_{ij} = \begin{cases} \dfrac{(-1)^{j-i}\binom{N}{i}\binom{N-i}{j-i}}{R^j}, & i \le j \\ 0, & i > j \end{cases}$$

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

On the other hand, the matrix form of the equation $y^2(x)$ is obtained as

$$y^2(x)=\begin{bmatrix}B_{0,N}(x) & B_{1,N}(x) & \cdots & B_{N,N}(x)\end{bmatrix}\begin{bmatrix}\mathbf{B}(x) & 0 & \cdots & 0\\ 0 & \mathbf{B}(x) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{B}(x)\end{bmatrix}\begin{bmatrix}y_0\mathbf{Y}\\ y_1\mathbf{Y}\\ \vdots\\ y_N\mathbf{Y}\end{bmatrix}$$

or briefly

$$y^2(x)=\mathbf{B}(x)\bar{\mathbf{B}}(x)\bar{\mathbf{Y}} \tag{6}$$

where

$$\bar{\mathbf{B}}(x)=\begin{bmatrix}\mathbf{B}(x) & 0 & \cdots & 0\\ 0 & \mathbf{B}(x) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{B}(x)\end{bmatrix}_{(N+1)\times(N+1)^2} \quad ,\bar{\mathbf{Y}}=\begin{bmatrix}y_0\mathbf{Y} & y_1\mathbf{Y} & \cdots & y_N\mathbf{Y}\end{bmatrix}^T_{(N+1)^2\times 1}.$$

By using the equations (4), (5) and (6) we obtain

$$y(x)y'(x)=\mathbf{B}(x)\bar{\mathbf{B}}(x)\bar{\mathbf{H}}\bar{\mathbf{Y}} \tag{7}$$

Following a similar way to (6), we have

$$\left(y'(x)\right)^2=\mathbf{B}(x)\mathbf{H}\bar{\mathbf{B}}(x)\bar{\mathbf{H}}\bar{\mathbf{Y}} \tag{8}$$

where

$$\bar{\mathbf{H}}=\begin{bmatrix}\mathbf{H} & 0 & \cdots & 0\\ 0 & \mathbf{H} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{H}\end{bmatrix}_{(N+1)^2\times(N+1)^2}.$$

3 Matrix relations based on collocation points

Let us use the collocation points defined by

$$x_i=a+\frac{b-a}{N}i \tag{9}$$

in order to

$$a=x_0<x_1<\cdots<x_N=b.$$

By using the collocation points (9) into Eq. (1), we obtain the system

$$P(x_i)y(x_i)+Q(x_i)y'(x_i)+R(x_i)y^2(x_i)+S(x_i)y(x_i)y'(x_i)+T(x_i)\left(y'(x_i)\right)^2=g(x_i),\ i=0,1,\ldots,N;\ 0\le x_i\le R; \tag{10}$$

By using the relations (4), (5), (6), (7) and (8); the system (10) can be written in the matrix form

$$\left[P(x_i)\mathbf{B}(x_i)\mathbf{I}+Q(x_i)\mathbf{B}(x_i)\mathbf{H}\right]\mathbf{Y}+\left[R(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)+S(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}+T(x_i)\mathbf{B}(x_i)\mathbf{H}\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}\right]\bar{\mathbf{Y}}=g(x_i) \quad (11)$$

Consequently, the fundamental matrix equations of (11) can be written in the following compact form

$$\mathbf{W}(x_i)\mathbf{Y}+\mathbf{V}(x_i)\bar{\mathbf{Y}}=g(x_i)$$

where

$$\mathbf{W}(x_i)=P(x_i)\mathbf{B}(x_i)\mathbf{I}+Q(x_i)\mathbf{B}(x_i)\mathbf{H}$$

and

$$\mathbf{V}(x_i)=R(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)+S(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}+T(x_i)\mathbf{B}(x_i)\mathbf{H}\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}.$$

Above expression can be rewritten shortly as

$$\mathbf{W}\mathbf{Y}^{*}+\mathbf{V}\bar{\mathbf{Y}}^{*}=\mathbf{G} \quad (12)$$

where

$$\mathbf{W}=\begin{bmatrix}\mathbf{W}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{W}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{W}(x_N)\end{bmatrix}_{(N+1)\times(N+1)^2},\quad \mathbf{V}=\begin{bmatrix}\mathbf{V}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{V}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{V}(x_N)\end{bmatrix}_{(N+1)\times(N+1)^3},$$

$$\mathbf{G}=\begin{bmatrix}g(x_0)\\ g(x_1)\\ \vdots\\ g(x_N)\end{bmatrix}_{(N+1)\times 1},\quad \mathbf{I}=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\mathbf{Y}^{*}=\begin{bmatrix}\mathbf{Y}\\ \mathbf{Y}\\ \vdots\\ \mathbf{Y}\end{bmatrix}_{(N+1)^2\times 1},\quad \bar{\mathbf{Y}}^{*}=\begin{bmatrix}\bar{\mathbf{Y}}\\ \bar{\mathbf{Y}}\\ \vdots\\ \bar{\mathbf{Y}}\end{bmatrix}_{(N+1)^3\times 1}.$$

4 Method of solution

The fundamental matrix equation (12) corresponding to Eq. (1) can be written as,

$$\mathbf{W}\mathbf{Y}^{*}+\mathbf{V}\bar{\mathbf{Y}}^{*}=\mathbf{G}$$

or

$$[\mathbf{W};\mathbf{V}:\mathbf{G}] \tag{13}$$

We can find the corresponding matrix equation for the condition (2), using the relation (4) and (6), as follows:

$$\left\{\alpha\mathbf{B}(a)+\beta\mathbf{B}(b)\right\}\mathbf{Y}+\left\{\gamma\mathbf{B}(a)\bar{\mathbf{B}}(a)+\tau\mathbf{B}(b)\bar{\mathbf{B}}(b)+\xi\mathbf{B}(a)\bar{\mathbf{B}}(b)\right\}\overline{\mathbf{Y}}=[\lambda], \tag{14}$$

so that

$$\mathbf{B}(a)=\left[B_{0,N}(a)\ B_{1,N}(a)\ \cdots\ B_{N,N}(a)\right],$$

$$\mathbf{B}(b)=\left[B_{0,N}(b)\ B_{1,N}(b)\ \cdots\ B_{N,N}(b)\right],$$

$$\bar{\mathbf{B}}(a)=\begin{bmatrix}\mathbf{B}(a) & 0 & \cdots & 0\\ 0 & \mathbf{B}(a) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{B}(a)\end{bmatrix}_{(N+1)\times(N+1)^2},\quad \bar{\mathbf{B}}(b)=\begin{bmatrix}\mathbf{B}(b) & 0 & \cdots & 0\\ 0 & \mathbf{B}(b) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{B}(b)\end{bmatrix}_{(N+1)\times(N+1)^2}.$$

We can write the corresponding matrix form (14) for the mixed condition (2) in the augmented matrix form as

$$[\mathbf{K};\mathbf{L}:\lambda], \tag{15}$$

where

$$\mathbf{K}=\left[k_0\ k_1\ \cdots\ k_N\right]_{1\times(N+1)}=\alpha\mathbf{B}(a)+\beta\mathbf{B}(b),$$

$$\mathbf{L}=\left[l_0\ l_1\ \cdots\ l_N\right]_{1\times(N+1)^2}=\gamma\mathbf{B}(a)\bar{\mathbf{B}}(a)+\tau\mathbf{B}(b)\bar{\mathbf{B}}(b)+\xi\mathbf{B}(a)\bar{\mathbf{B}}(b).$$

To obtain the approximate solution of Eq. (1) with the mixed condition (2) in the terms of Bernstein polynomials, by replacing the row matrix (15) by the last row of the matrix (12), we obtain the required augmented matrix:

$$[\tilde{\mathbf{W}};\tilde{\mathbf{V}}:\tilde{\mathbf{G}}]=\left[\begin{array}{cccc;cccc:c}\mathbf{W}(x_0) & 0 & \cdots & 0 & \mathbf{V}(x_0) & 0 & \cdots & 0 & g(x_0)\\ 0 & \mathbf{W}(x_1) & \cdots & 0 & 0 & \mathbf{V}(x_1) & \cdots & 0 & g(x_1)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \mathbf{K} & 0 & 0 & \cdots & \mathbf{L} & \lambda\end{array}\right]$$

or the corresponding matrix equation

$$\tilde{\mathbf{W}}\mathbf{Y}^{*}+\tilde{\mathbf{V}}\bar{\mathbf{Y}}^{*}=\tilde{\mathbf{G}} \tag{16}$$

where

$$\tilde{\mathbf{W}}=\begin{bmatrix}\mathbf{W}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{W}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{K}\end{bmatrix},\quad \tilde{\mathbf{V}}=\begin{bmatrix}\mathbf{V}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{V}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{L}\end{bmatrix},\quad \tilde{\mathbf{G}}=\begin{bmatrix}g(x_0)\\ g(x_1)\\ \vdots\\ \lambda\end{bmatrix}.$$

The unknown coefficients $\{y_0, y_1, \cdots, y_N\}$ can be determined from the nonlinear system (16). As a result, we can obtain approximate solution in the truncated series form (3).

5 Accuracy of solution

We can check the accuracy of the method. The truncated Benstein series in (3) have to be approximately satisfying Eq. (1). For each $x = x_i \in [0, R]$, $i = 0, 1, \ldots, N$

$$E(x_i) = \left| P(x_i)\, y(x_i) + Q(x_i)\, y'(x_i) + R(x_i)\, y^2(x_i) + S(x_i)\, y(x_i)\, y'(x_i) + T(x_i) \left(y'(x_i) \right)^2 - g(x_i) \right| \cong 0$$

and $E(x_i) \le 10^{-k_i}$ (k_i is any positive integer).

If $\max\left(10^{-k_i}\right) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit $N$is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k} [9-13].

6 Numerical examples

In this section, three numerical examples are given to illustrate the accuracy and efficiency of the presented method.

Example 6.1. Let us first consider the nonlinear differential equation

$$3y^2(x) - xy'(x) - 3y(x) = 3x^6 \tag{17}$$

with the non-linear condition $y^2(0) + y(0) = 2$, $0 \le x \le 1$.

The approximate solution $y(x)$ by the truncated Bernstein polynomial, $y(x) = \sum_{n=0}^{3} y_n B_{n,3}(x)$, where $0 \le x \le 1$, $P(x) = -3$, $Q(x) = -x$, $R(x) = 3$, $S(x) = T(x) = 0$, $g(x) = 3x^6$. For $N = 3$ the collocation points become,

$$x_0 = 0\ ,\ x_1 = \frac{1}{3}\ ,\ x_2 = \frac{2}{3}\ ,\ x_3 = 1.$$

From the fundamental matrix equations for the given equation and condition respectively are obtained as,

$$\left[P(x_i)\, \mathbf{B}(x_i)\, \mathbf{I} + Q(x_i)\, \mathbf{B}(x_i)\, \mathbf{H} \right] \mathbf{Y} + \left[R(x_i)\, \mathbf{B}(x_i)\, \bar{\mathbf{B}}(x_i) \right] \bar{\mathbf{Y}} = g(x_i)\,.$$

We can find the matrix equations from,

$$\mathbf{W}(x_i)\, \mathbf{Y} + \mathbf{V}(x_i)\, \bar{\mathbf{Y}} = g(x_i)\,.$$

The fundamental matrix equation

$$\mathbf{W}\mathbf{Y}^{*} + \mathbf{V}\bar{\mathbf{Y}}^{*} = \mathbf{G}$$

where

$$\mathbf{W}=\begin{bmatrix}\mathbf{W}(0) & 0 & 0 & 0\\ 0 & \mathbf{W}\left(\frac{1}{3}\right) & 0 & 0\\ 0 & 0 & \mathbf{W}\left(\frac{2}{3}\right) & 0\\ 0 & 0 & 0 & \mathbf{W}(1)\end{bmatrix},\ \mathbf{V}=\begin{bmatrix}\mathbf{V}(0) & 0 & 0 & 0\\ 0 & \mathbf{V}\left(\frac{1}{3}\right) & 0 & 0\\ 0 & 0 & \mathbf{V}\left(\frac{2}{3}\right) & 0\\ 0 & 0 & 0 & \mathbf{V}(1)\end{bmatrix},$$

$$\mathbf{G}=\left[g(0)\ g\left(\tfrac{1}{3}\right)\ g\left(\tfrac{2}{3}\right)\ g(1)\right]^T.$$

The matrix forms of the conditions are

$$\left\{\mathbf{B}(0)\bar{\mathbf{B}}(0)\right\}\bar{\mathbf{Y}}+\mathbf{B}(0)\mathbf{Y}=2.$$

The augmented matrix for this fundamental matrix equation is calculated as

$$[\tilde{\mathbf{W}},\tilde{\mathbf{V}}:\tilde{\mathbf{G}}]=\begin{bmatrix}\mathbf{W}(0) & 0 & 0 & 0, & \mathbf{V}(0) & 0 & 0 & 0: & g(0)\\ 0 & \mathbf{W}\left(\frac{1}{3}\right) & 0 & 0, & 0 & \mathbf{V}\left(\frac{1}{3}\right) & 0 & 0: & g\left(\frac{1}{3}\right)\\ 0 & 0 & \mathbf{W}\left(\frac{2}{3}\right) & 0, & 0 & 0 & \mathbf{V}\left(\frac{2}{3}\right) & 0: & g\left(\frac{2}{3}\right)\\ 0 & 0 & 0 & \mathbf{K}, & 0 & 0 & 0 & \mathbf{L}: & \lambda\end{bmatrix}$$

where

$$\mathbf{W}(0)=\left[-3\ 0\ 0\ 0\right],\mathbf{W}\left(\frac{1}{3}\right)=\left[-\tfrac{4}{9}\ -\tfrac{4}{3}\ -1\ -\tfrac{2}{9}\right],\mathbf{W}\left(\frac{2}{3}\right)=\left[\tfrac{1}{9}\ 0\ -\tfrac{4}{3}\ -\tfrac{16}{9}\right],$$

$$\mathbf{V}(0)=\left[3\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right],$$

$$\mathbf{V}\left(\frac{1}{3}\right)=\left[\tfrac{64}{243}\ \tfrac{32}{81}\ \tfrac{16}{81}\ \tfrac{8}{243}\ \tfrac{32}{81}\ \tfrac{16}{27}\ \tfrac{8}{27}\ \tfrac{4}{81}\ \tfrac{16}{81}\ \tfrac{8}{27}\ \tfrac{4}{27}\ \tfrac{2}{81}\ \tfrac{8}{243}\ \tfrac{4}{81}\ \tfrac{2}{81}\ \tfrac{1}{243}\right],$$

$$\mathbf{V}\left(\frac{2}{3}\right)=\left[\tfrac{1}{243}\ \tfrac{2}{81}\ \tfrac{4}{81}\ \tfrac{8}{243}\ \tfrac{2}{81}\ \tfrac{4}{27}\ \tfrac{8}{27}\ \tfrac{16}{81}\ \tfrac{4}{81}\ \tfrac{8}{27}\ \tfrac{16}{27}\ \tfrac{32}{81}\ \tfrac{8}{243}\ \tfrac{16}{81}\ \tfrac{32}{81}\ \tfrac{64}{243}\right],$$

$$\mathbf{K}=\left[1\ 0\ 0\ 0\right],g(0)=0,g\left(\frac{1}{3}\right)=\frac{1}{243},g\left(\frac{2}{3}\right)=\frac{64}{243},$$

$$\mathbf{L}=\left[1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right],\lambda=2,$$

From the obtained system, the coefficients y_0,y_1,y_2 and y_3 are found as $y_0=1,y_1=1,y_2=1$and $y_3=0$.

Therefore, we have the Bernstein polynomial solution,

$$y(x)=1-x^3$$

which is an exact solution.

Example 6.2. Consider the following nonlinear differential equation

$$\left(y'(x)\right)^2+3xy'(x)-y(x)=-\frac{3}{2}x^2 \tag{18}$$

with the non-linear initial condition $2y(0) - y^2(0) = 1$, $0 \le x \le 1$. So that $P(x) = -1$, $Q(x) = 3x$, $T(x) = 1$, $R(x) = S(x) = 0$, $g(x) = -\frac{3}{2}x^2$. For $N = 3$ the collocation points become,

$$x_0 = 0\ ,\ x_1 = \frac{1}{3}\ ,\ x_2 = \frac{2}{3}\ ,\ x_3 = 1.$$

From the fundamental matrix equations for the given equation and condition respectively are obtained as,

$$\left[P(x_i)\mathbf{B}(x_i)\mathbf{I} + Q(x_i)\mathbf{B}(x_i)\mathbf{H}\right]\mathbf{Y} + \left[T(x_i)\mathbf{B}(x_i)\mathbf{H}\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}\right]\bar{\mathbf{Y}} = g(x_i)$$

We can find the compact form of this matrix equations from

$$\mathbf{W}(x_i)\mathbf{Y} + \mathbf{V}(x_i)\bar{\mathbf{Y}} = g(x_i)$$

The fundamental matrix equation,

$$\mathbf{W}\mathbf{Y}^* + \mathbf{V}\bar{\mathbf{Y}}^* = \mathbf{G}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}(0) & 0 & 0 & 0 \\ 0 & \mathbf{W}\left(\frac{1}{3}\right) & 0 & 0 \\ 0 & 0 & \mathbf{W}\left(\frac{2}{3}\right) & 0 \\ 0 & 0 & 0 & \mathbf{W}(1) \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \mathbf{V}(0) & 0 & 0 & 0 \\ 0 & \mathbf{V}\left(\frac{1}{3}\right) & 0 & 0 \\ 0 & 0 & \mathbf{V}\left(\frac{2}{3}\right) & 0 \\ 0 & 0 & 0 & \mathbf{V}(1) \end{bmatrix},$$

$$\mathbf{G} = \left[g(0)\ g\left(\tfrac{1}{3}\right)\ g\left(\tfrac{2}{3}\right)\ g(1)\right]^T.$$

The matrix forms of the conditions are

$$2\mathbf{B}(0)\mathbf{Y} - \left\{\mathbf{B}(0)\bar{\mathbf{B}}(0)\right\}\bar{\mathbf{Y}} = 1.$$

The augmented matrix for this fundamental matrix equation is calculated as

$$\left[\tilde{\mathbf{W}}, \tilde{\mathbf{V}} : \tilde{\mathbf{G}}\right] = \begin{bmatrix} \mathbf{W}(0) & 0 & 0 & 0, & \mathbf{V}(0) & 0 & 0 & 0: & g(0) \\ 0 & \mathbf{W}\left(\frac{1}{3}\right) & 0 & 0, & 0 & \mathbf{V}\left(\frac{1}{3}\right) & 0 & 0: & g\left(\frac{1}{3}\right) \\ 0 & 0 & \mathbf{W}\left(\frac{2}{3}\right) & 0, & 0 & 0 & \mathbf{V}\left(\frac{2}{3}\right) & 0: & g\left(\frac{2}{3}\right) \\ 0 & 0 & 0 & \mathbf{K}, & 0 & 0 & 0 & \mathbf{L}: & \lambda \end{bmatrix}$$

where

$$\mathbf{W}(0) = \left[-1\ 0\ 0\ 0\right], \mathbf{W}\left(\frac{1}{3}\right) = \left[-\tfrac{44}{27}\ -\tfrac{4}{9}\ \tfrac{7}{9}\ \tfrac{8}{27}\right], \mathbf{W}\left(\frac{2}{3}\right) = \left[-\tfrac{19}{27}\ -\tfrac{20}{9}\ -\tfrac{4}{9}\ \tfrac{64}{27}\right],$$

$$\mathbf{V}(0) = \left[9\ \ -9\ \ 0\ \ 0\ \ -9\ \ 9\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\right],$$

$$\mathbf{V}\left(\frac{1}{3}\right) = \left[\tfrac{16}{9}\ 0\ -\tfrac{4}{3}\ -\tfrac{4}{9}\ 0\ \ 0\ \ 0\ \ 0\ -\tfrac{4}{3}\ 0\ 1\ \tfrac{1}{3}\ -\tfrac{4}{9}\ 0\ \tfrac{1}{3}\ \tfrac{1}{9}\right],$$

$$\mathbf{V}\left(\frac{2}{3}\right)=\left[\frac{1}{9}\ \frac{1}{3}\ 0\ -\frac{4}{9}\ \frac{1}{3}\ 1\ 0\ -\frac{4}{3}\ 0\ 0\ 0\ 0\ -\frac{4}{9}\ -\frac{4}{3}\ 0\ \frac{16}{9}\right],$$

$$\mathbf{K}=\left[2\ \ 0\ \ 0\ \ 0\right], g\left(0\right)=0, g\left(\frac{1}{3}\right)=-\frac{1}{6}, g\left(\frac{2}{3}\right)=-\frac{2}{3}, \lambda=1,$$

$$\mathbf{L}=\left[1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\right],$$

From the obtained system, the coefficients y_0, y_1, y_2 and y_3 are found as $y_0=1, y_1=\frac{4}{3}, y_2=\frac{3}{2}$and $y_3=\frac{3}{2}$. Hence we have the Bernstein polynomial solution

$$y\left(x\right)=-\frac{1}{2}x^2+x+1.$$

Example 6.3. Lets consider following nonlinear differential equation

$$x^2\left(y'\left(x\right)\right)^2+y'\left(x\right)-x^2y^2\left(x\right)=e^x \tag{19}$$

with the non-linear initial condition $y\left(0\right)+3y^2\left(0\right)=4$, $0\le x\le 1$. So that

$$Q\left(x\right)=1\ ,\ R\left(x\right)=-x^2\ ,\ T\left(x\right)=x^2\ , P\left(x\right)=S\left(x\right)=0,\ g\left(x\right)=e^x.$$

The solutions obtained for $N=2,5,7$ are compared with the exact solution is e^x, which are given in Figure 1. We compare the numerical solution and absolute errors for $N=2,5,7$ in Table 1.

Table 1. Comparison of the numerical errors of Example 6.3.

		Numerical Solutions			Errors		
x_i	Exact Solution	$N=2$	$N=5$	$N=7$	$N=2$	$N=5$	$N=7$
0	1	1	1	1	0	0	0
0.1	1.105170918	1.105	1.105170917	1.105170918	1.70918E-4	1E-9	0
0.2	1.221402758	1.22	1.221402667	1.221402758	1.40275E-3	9.1E-8	0
0.3	1.349858807	1.345	1.34985775	1.349858806	4.85880E-3	1.058E-6	2E-9
0.4	1.491824697	1.48	1.491818667	1.491824681	1.11824E-2	6.031E-6	1.7E-8
0.5	1.648721271	1.625	1.648697917	1.648721168	2.37212E-2	2.335E-5	1.03E-7
0.6	1.822118801	1.78	1.822048	1.822118354	4.21188E-2	7.08E-5	4.46E-7
0.7	2.013752707	1.945	2.013571417	2.013751158	6.87527E-2	1.812E-4	1.549E-6
0.8	2.225540928	2.12	2.225130667	2.225536366	1.0554E-1	4.102E-4	4.562E-6
0.9	2.459603111	2.305	2.45875825	2.459591263	1.54603E-1	8.448E-4	1.184E-5
1	2.718281828	2.5	2.716666667	2.718253968	2.18281E-1	1.615E-3	2.786E-5

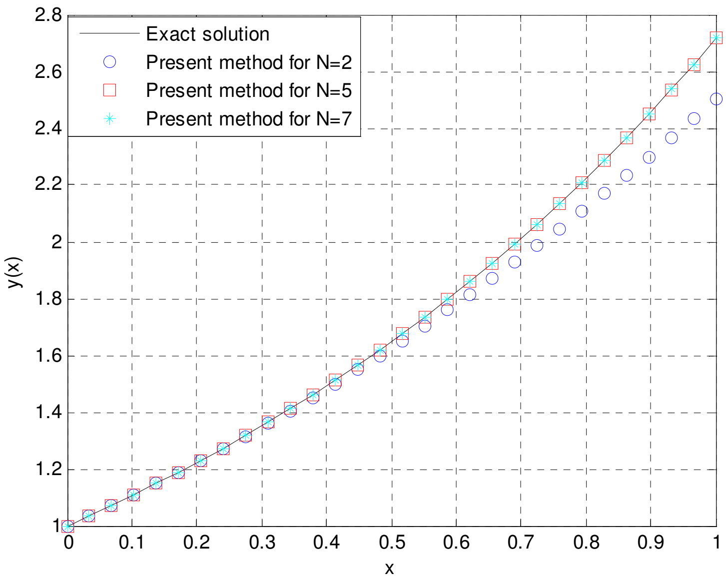

Fig. 1. Numerical and exact solution of Example 6.3 for $N = 2,5,7$.

7 Conclusion

A new technique, using the Bernstein polynomial, to numerically solve the first order nonlinear differential equations is presented. Nonlinear differential equations are usually difficult to solve analyticaly. Then it is required to obtain the approximate solutions. For this reason, the present method has been proposed for approximate solution and also analytical solution.

In this paper we have presented a suggested method to solve second order nonlinear ordinary differential equations with mixed non-linear conditions using the matrix method based on collocation points on any interval $[0,R]$. The matrix method avoids the difficulties and massive computational work by determining the analytic solution.

On the other hand, from Table 1, it may be observed that the errors found for different N show close agreement for various values of x_i. Table and Figure indicate that as N increases, the errors decrease more rapidly; hence for better results, using large number N is recommended. A considerable advantage of the method is that Bernstein coefficients of the solution are found very easily by using the computer programs. On the other hand our Nth order approximation gives the exact solution when the solution is polynomial of degree equal to or less than N. If the solution is not polynomial, Bernstein series approximation converges to the exact solution as N increases.

The method can also be extended to the high order nonlinear differential equations with variable coefficients, but some modifications are required.

References

[1] S.N. Bernstein, Demonstration du théoréme de Weierstrass fondée sur la calcul des probabilites, Communications of the Mathematical Society of Charkow. 2 (13) (1912-1913) 1-2.

[2] K.I. Joy, Bernstein polynomials, On-Line Geometric Modeling Notes, http://en.wikipedia.org/wiki/Bernstein|-polynomial.

[3] G.F.Corliss, Guarented error bounds for ordinary differential equations, in theory and numeric of ordinary and partial equations (M.Ainsworth, J.Levesley, W.A Light, M.Marletta, Eds), Oxford Universty press, Oxford,(1995), pp.342.

[4] A. Güner, S. Yalçınbaş, "Legendre collocation method for solving nonlinear differential equations", Mathematical &Computational Applications, 18 (3) (2013) 521-530.

[5] D. Taştekin, S. Yalçınbaş, M. Sezer, "Taylor collocation method for solving a class of the first order nonlinear differential equations", Mathematical &Computational Applications, 18(3) (2013) 383-391.

[6] S. Yalçınbaş, D.Taştekin, "Fermat collocation method for solving a class of the second order nonlinear differential equations", Applied Mathematics and Physics, 2 (2) (2014)33-39.

[7] D. Taştekin, S. Yalçınbaş, "Fermat collocation method for solving a class of the first order nonlinear differential equations", Journal of Mathematical Sciences and Applications, 2 (1) (2014) 4-9.

[8] S. Yalçınbaş, K. Erdem Biçer, D. Taştekin, "Fermat collocation method for nonlinear system of second order boundary value problems", Kasmera Journal, 43 (1) (2015) 53-66.

[9] M. Sezer, A method for the approximate solution of the second order linear differential equations in terms of Taylor polynomials, International Journal of Mathematical Education in Science and Technology 27(6) (1996) 821-834.

[10] D. Taştekin, S. Yalçınbaş, M. Sezer, Taylor collocation method for solving a class of the first order nonlinear differential equations, Mathematical and Computational Applications, 18 (3)(2013) 383-391.

[11] S. Yalçınbaş, Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations, Applied Mathematics and Computation 127 (2002) 195-206.

[12] S. Yalçınbaş, K. Erdem, Approximate solutions of nonlinear Volterra integral equation systems, International Journal of Modern Physics B 24 (32) (2010) 6235-6258.

[13] S. Yalçınbaş, K. Erdem, "A New Approximation Method for the Systems of Nonlinear Fredholm Integral Equations", Applied Mathematics and Physics, 2 (2) (2014) 40-48.