On classification of some finite linear spaces

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Abstract: We classified all finite linear spaces having property: for every pair L, L' of intersecting lines, and every point p not on either of these, there are ℓ lines through p disjoint from both L and L', where the nonnegative number ℓ is constant. This space will be denoted by $(2_i, 0, \ell)$-interlaced.

In the present paper, we will show that this property (with ℓ arbitrary) essentially characterizes the semiaffine planes with non-constant line size. For linear spaces with constant line size, this property does not seem to be strong enough, but we derive some necessary conditions on the parameters showing that examples will be hard to find.

Keywords: Linear space, interlaced space, projective plane, affine plane, semiaffine linear space.

1 Introduction

Linear spaces lie at the foundation of incidence geometry, and more in particular, of finite geometry. A lot of characterizations of projective and affine spaces use linear spaces. Also, many important diagram geometries related to classes of simple groups are build with linear spaces. Linear spaces with constant block size are called Steiner systems and also play a prominent role in finite geometry. But there are also linear spaces that are not Steiner systems, and yet they appear often naturally. One such class of linear spaces is the class of semiaffine planes. The definition of such a structure uses an inequality (namely, for each line L and each point p not on L, there are ≤ 1 lines through p not meeting L). Not surprisingly, this inequality in the definition gives rise to examples with different line sizes, as follows from Dembowski's classification. Yet, these structures have common properties that can be stated without an inequality:

i. for every pair L, L' of intersecting lines, and every point p not on either of these, there are ℓ lines through p disjoint from both L and L', where the nonnegative number ℓ is constant.

In the present paper, we will show that this property (with ℓ arbitrary) essentially characterizes the semiaffine planes with non-constant line size. For linear spaces with constant line size, this property does not seem to be strong enough, but we derive some necessary conditions on the parameters showing that examples will be hard to find. We now get down to more precise definitions and statements.

2 Definitions and statements

A *linear space* $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ consists of a *point set* $\mathscr{P}$ and a line set $\mathscr{L}$, which is a subset of the power set of P (the *power set* of a set is the family of subsets of it), with the property that every two points are contained in exactly one line,

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and such that every line contains at least two, but not all, points. Linear spaces can be viewed as incidence structures, and we will use the special terminology of these structures. For instance, we will say that a line goes through a point if the point is contained in the line, or that two lines are concurrent or intersecting if they intersect nontrivially.

A linear space $\mathscr{S}$ is called *finite* if it has only a finite number of points. The unique line passing through two distinct points x and y will be denoted by xy. The *degree* $\delta(p)$ of a point p is the number of lines through p. The definition of a linear space easily implies that $\delta(p) > 1$, for every point of a linear space. If in a finite linear space all lines have the same size, then all points have the same degree. In such a case we usually denote the line size by $s+1$ and the point degree by $t+1$ and say that $\mathscr{S}$ has order (s,t). A linear space of order (s,t) is a $(2,s+1,st+s+1)$-Steiner system. Conversely, a $(2,k,v)$-Steiner system is by definition a linear space of order $(k-1,\frac{v-k}{k-1})$. In an arbitrary finite linear space $\mathscr{S}$, we will denote by $|L|$ the size of a line L.

A *semiaffine plane* $\mathscr{S} = (P,\mathscr{L})$ is a linear space with the property that every pair of intersecting lines meets every other line nontrivially. Dembowski showed in 1962 that a finite semiaffine plane is either a projective plane, an affine plane, a punctured projective plane (i.e., a projective plane with one point removed), or an affine plane with one point at infinity added (i.e., a projective plane with one line and all but one points of that line removed), see [4].

A *complete graph* $\mathscr{S}$ is a linear space with all lines of size 2. A *partial linear space* $\mathscr{S} = (P,\mathscr{L})$ consists of a set P of points and a set $\mathscr{L}$ of subsets of P with the property that every two points are contained in at most one line, and such that every line contains at least two, but not all, points. It is a *partial geometry (with parameters* (s,t,α)*)* if every line has exactly $s+1$ points, every point is contained in exactly $t+1$ lines, and for every point x and every line $L \not\ni x$ there are exactly α lines containing x and meeting L in a point. A *symmetric* 2-(v,k,λ) *design* $\mathscr{S} = (P,\mathscr{L})$ consists of a set P of v points and a set $\mathscr{L}$ of v subsets of P, called blocks, with the property that every block contains exactly k points and two points are contained in exactly λ blocks (with $v > k > \lambda > 0$). Standard counting arguments imply that every point is contained in exactly k blocks and that two blocks meet in exactly λ points.

A *configuration* $\mathfrak{C}$ consists of a set P' of points, together with a set $\mathscr{L}'$ of lines such that every line is a subset of P', and every point is in at least one line. Two configurations are *equivalent* if there is a bijection β between their line sets inducing a bijection between the set of points contained in at least two lines. Let $\mathscr{L}'$ be a set of lines of a linear space $\mathscr{S}$, then $\mathscr{L}'$ together with all points of $\mathscr{S}$ in the union of these lines, define a configuration $\mathfrak{C}$ which we shall call *induced by* $\mathscr{L}'$.

Now let $\mathscr{S}$ be a linear space, and let $\mathfrak{C}$ be isomorphic to a configuration induced by some line set of $\mathscr{S}$. Then we say that $\mathscr{S}$ is $(\mathfrak{C},k,\ell)$*-interlaced* if for every line set $\mathscr{L}'$ inducing a configuration equivalent to $\mathfrak{C}$, and every point p not in the union of $\mathscr{L}'$, there are precisely ℓ lines of $\mathscr{S}$ through p meeting the union of $\mathscr{L}'$ in precisely k points. If $\mathfrak{C}$ consists of a single line, then $(\mathfrak{C},k,\ell)$-interlaced will be written as $(1,k,\ell)$-interlaced, while $(2_i,k.\ell)$-interlacing will mean $(\mathfrak{C},k,\ell)$-interlacing for $\mathfrak{C}$ two intersecting lines (the intersection being one single point in the latter case). Our main results are the following.

Theorem 1. *If $\mathscr{S}$ is a finite $(2_i,0,\ell)$-interlaced linear space, then either $\mathscr{S}$ is a Steiner system, or $\mathscr{S}$ is a semiaffine plane with non-constant line size.*

This immediately implies the following characterizations of classes semiaffine planes:

Corollary 1. *A finite linear space $\mathscr{S}$ with non-constant line size is a semiaffime plane if and only if it is $(2_i, 0, \ell)$-interlaced for some nonnegative integer ℓ.*

We will use the following notation.

The number of lines passing through a point x and intersecting a subset $\mathscr{B}$ of points in exactly k distinct points will be denoted by $i_k(x, \cup\{l : l \in \mathscr{B}\})$.

Given a triple (x, L, M) where L and M are two intersecting lines and $x \notin L \cup M$, the number of lines passing through x and intersecting L but not M will be denoted by $i_1(x, L, M)$.

In the next section, we prove our main results. In Section 4, we will prove some other general properties of $(1, k, \ell)$-interlaced, $(2_i, k, \ell)$-interlaced and $(2_d, k, \ell)$-interlaced linear spaces.

3 Proof of the main results

3.1 Theorem A

We start by proving Theorem A. First we deal with the the small cases, then we treat the case of non-constant line size, and finally we look at the case of constant line size.

Note that, whenever a linear space is $(2_i, 0, 0)$-interlaced, it is a semiaffine plane.

Lemma 1. If $\mathscr{S}$ is a $(2_i, 0, \ell)$-interlaced linear space containing at least one point with degree 2, then $\mathscr{S}$ is a semiaffine plane.

Proof. Let p be a point with $\delta(p) = 2$. If there is only one line not through p, then we have a complete graph on 3 points, hence a semiaffine plane. In the other cases we clearly see that $\ell = 0$, and hence we again have a semiaffine plane.

Lemma 2. If $\mathscr{S}$ is a $(2_i, 0, \ell)$-interlaced space containing at least one point of degree 3, then $\mathscr{S}$ is a semiaffine plane.

Proof. Let p be a point with $\delta(p) = 3$ and let A, B, C be three distinct lines passing through p. If $t \neq 0$, then is at least one line D disjoint from $B \cup C$ and passing through a point $x \neq p$ of the line A. Now D contains at least two points x, y, and the line py is different from A, B, C. Hence $\delta(p) \geq 4$, a contradiction.

Hence, from now on, we may assume that every point is contained in at least four lines. However, not all our lemmas will need this restriction, and we will clearly indicate which one doe and which ones do not.

The following lemma is a fundamental observation.

Lemma 3. Let $\mathscr{S}$ be a $(2_i,0,\ell)$-interlaced space and let A,B,C be three pairwise intersecting lines of $\mathscr{S}$.

i. If $A\cap B\cap C=\emptyset$, then

$$\sum_{p\in A}\delta(p)-\sum_{q\in B}\delta(q)=(|A|-|B|)\,(|C|+\ell)+\delta(A\cap B)-\delta(B\cap C).$$

ii. If $A\cap B\cap C\neq\emptyset$, then

$$\sum_{p\in A}\delta(p)-\sum_{q\in B}\delta(q)=(|A|-|B|)\,(|C|+\ell)\ .$$

Proof. We show (i). For any point $x\in A\setminus(B\cup C)$ and $y\in B-\setminus(A\cup C)$, it is clear, by definition, that

$$i_1(x,B,C)=\delta(x)-|C|-\ell \text{ and } i_1(y,A,C)=\delta(y)-|C|-\ell.$$

Therefore,

$$\sum_{x\in A\setminus(B\cup C)} i_1(x,B,C)=\left(\sum_{x\in A\setminus-(B\cup C)}\delta(x)\right)-(|C|+\ell)\,(|A|-2)\,, \tag{1}$$

$$\sum_{y\in B\setminus(A\cup C)} i_1(y,A,C)=\left(\sum_{y\in B\setminus(A\cup C)}\delta(y)\right)-(|C|+\ell)\,(|B|-2)\,. \tag{2}$$

Now, clearly,

$$\sum_{p\in A}\delta(p)=\delta(A\cap C)+\delta(A\cap B)+\sum_{p\in A\setminus(B\cup C)}\delta(p), \tag{3}$$

$$\sum_{q\in B}\delta(q)=\delta(B\cap C)+\delta(A\cap B)+\sum_{q\in B\setminus(A\cup C)}\delta(q). \tag{4}$$

Counting in two ways the number of lines, disjoint from C, and intersecting A and B in two distinct points, we obtain

$$\sum_{x\in A\setminus(B\cup C)} i_1(x,B,C)=\sum_{y\in B\setminus(A\cup C)} i_1(y,A,C). \tag{5}$$

Combining the equalities (1) up to (5), we obtain (i). The second case, (ii), is proved completely similarly.

Lemma 4. Let $\mathscr{S}$ be a $(2_i,0,\ell)$-interlaced linear space, and $b(x)\geq 3$ for each point x of $\mathscr{S}$. If A and B are two intersecting distinct lines of the same size, then $\sum_{p\in A}\delta(p)=\sum_{q\in B}\delta(q)$.

Proof. Let A and B be two intersecting lines with $|A|=|B|$. Since $\delta(x)\geq 3$ for each point x of S, there is a line C concurrent with A and B (meaning $A\cap\ B\cap C\neq\emptyset$). By Lemma 3.1$(ii)$,

$$\sum_{p\in A}\delta(p)-\sum_{q\in B}\delta(q)=(|A|-|B|)\,(|C|+\ell)=0.$$

This proves the lemma.

Lemma 5. If $\mathscr{S}$ is a $(2_i,0,\ell)$-interlaced linear space, and $\delta(x) \geq 4$ for each point x of $\mathscr{S}$, then, for each point x, there are at least $\delta(x)-1$ lines passing through x having the same size.

Proof Let p be a point of $\mathscr{S}$. If all lines through p have the same size, then there is nothing to prove. Hence we may assume that there are two lines A,B through p with $|A| \neq |B|$. Let C be any other line through p. Since by hypothesis $\delta(p) \geq 4$, we may take a line D through p with $D \notin \{A,B,C\}$. Now, by Lemma 3.1(ii),

$$\sum_{p\in A}\delta(p)-\sum_{q\in B}\delta(q)=(|A|-|B|)(|C|+\ell)=(|A|-|B|)(|D|+\ell).$$

Therefore, $|C|=|D|$. If $|B|=|C|$, then the lemma is again proved. But if $|B|\neq|C|$, then interchanging the roles of A and C, we obtain with the same argument that $|A|=|D|=|C|$, and the lemma again follows.

Lemma 6. If $\mathscr{S}$ is a $(2,0,\ell)$-interlaced linear space and $\delta(x) \geq 3$ for each point x of $\mathscr{S}$, then all points of any two concurrent lines of the same size, except possibly the intersection point, have the same degree.

Proof. Let A and B be lines passing through a point p and let $q \in A-\{p\}$, $r \in B-\{p\}$. Put $C=qr$. By Lemma 3.1, $\sum_{x\in A}\delta(x)=\sum_{y\in B}\delta(y)$. Since $|A|=|B|$, and since $A\cap C=\{x\}$ and $B\cap C=\{y\}$, Lemma 3.1(i) implies $\delta((x)=\delta(y)$.

Lemma 7. Let $\mathscr{S}$ be a $(2,0,\ell)$-interlaced linear space with $\delta(x)\geq 4$ for each point x of $\mathscr{S}$. Let B be a line passing through a point p. If all lines passing through p except for B have the same size (different from $|B|$), then

i. any two points outside B have the same degree, and
ii. any two lines distinct from B but concurrent with B have the same size.

Proof. Let x and y be two points outside B such that x,y,p are non-collinear. By Lemma 3.1, all lines distinct from B through p have the same size n, and so we deduce from Lemma 3.1 that x and y have the same degree. This implies that all points outside B have the same degree r. This proves (i).

Let L and M be two lines distinct from B and intersecting B in a common point q.

If $q=p$, we already know that L and M have the same size, so we may assume that $q\neq p$.

Let A be a line through p, distinct from B and intersecting L and let A' be a line through p, distinct from B and intersecting M. Then, by Lemma 3.1 and lemma 3.1, we obtain

$\sum_{x\in A}\delta(x)-\sum_{y\in B}\delta(y)=(|A|-|B|)(|L|+\ell)+r-\delta(q)$ and
$\sum_{x\in A}\delta(x)-\sum_{y\in B}\delta(y)=(|A|-|B|)(|M|+\ell)+r-\delta(q)$.

Since $|A|\neq|B|$, these equalities imply $|L|=|M|$. Therefore, (ii) is proved.

Lemma 8. Let $\mathscr{S} = (P, \mathscr{L})$ be a $(2_i, 0, \ell)$-interlaced linear space, with $\delta(x) \geq 4$ for each point x of $\mathscr{S}$. Let p be a point of $\mathscr{S}$ such that all lines through p have the same size n, except for one line B, which has size different from n. Then there is at least one line of size n intersecting B and not passing through p.

Proof. Suppose by way of contradiction that every line intersecting B and not passing through p has size distinct from n. If $|B| > 2$, any point $x \notin B$ is on at least two lines of size distinct from n. Therefore, by Lemma 3.1, the line xp is the only line passing through$x$ having size n. From Lemma 3.1(ii), we deduce that any two lines distinct from xp and concurrent with xp have the same size. Since $\delta(p) \geq 4$ and $p \in px$, this contradicts the fact that B is the only line of size $|B|$ containing p. This show that $|B| = 2$ and also that any point off the line B is on at least two lines of size n. We may set $B = \{p, q\}$. By hypothesis, the size of the line xq is distinct from n. Thus,we deduce from Lemma 3.1 that the line xq is the only line of size $|xq|$ containing x. By Lemma 3.1(ii), every point y outside xq has degree $\delta(p)$. Therefore,

$$\delta(y) = \delta(p), \tag{1}$$

for all points $y \neq q$.

Let v denote the number of all points of $mathcalS$. Since the line B is the only line of size $2 \neq n$ passing through p, we have

$$v - 1 = (\delta(p) - 1)(n-1) + 1, \tag{2}$$

and since the line yq is the only line of size $|yq| \neq n$ containing y, we have

$$v - 1 = (\delta(y) - 1)(n-1) + (|yq| - 1). \tag{3}$$

We now easily see that the equalities (1) up to (3) imply $|yq| = 2$. Therefore, all lines containing the point q have the same size 2 and the other lines of $\mathscr{S}$ all have size n. Thus, $\mathscr{S}' = (P \setminus -\{q\}$ is a $(2, n, v-1)$ Steiner system in which every point has degree $\frac{v-2}{n-1}$.

Let L be a line through p distinct from B and let M be a line through q distinct from B, but meeting L in a unique point. If $u \in L \setminus (B \cup M)$, then $\ell = i_0(u, B \cup M) = \frac{v-2}{n-1} - 1$. But if w is a point outside $L \cup B \cup M$ (which exists since $\delta p \geq 4$), then $\ell = i_0(w, B \cup M) = \frac{v-2}{n-1} - 2$, a contradiction. This completes the proof of the lemma.

Lemma 9. Let $\mathscr{S}$ be a $(2_i, 0, \ell)$-interlaced linear space with $\delta(x) \geq 4$ for each point x of $\mathscr{S}$. Let A and B be two lines with distinct sizes passing through a point p, and suppose B is the unique line through p of size $|B|$. Then either the lines of size $|B|$ are pairwise disjoint, or they all meet in a common point r.

Proof. By Lemma 3.1, there is a line intersecting B not passing through p and having size $|A|$. By Lemma 3.1(ii), there is a line C intersecting A and B in two distinct points and having size $|A|$. By Lemma 3.1(i),

$$\sum_{x \in A} \delta(x) - \sum_{y \in B} \delta(y) = (|A| - |B|)(|A| + \ell) + \delta(A \cap C) - \delta(B \cap C). \tag{1}$$

By Lemma 3.1(i), the common degree of points outside B is $\delta(A\cap C)$.

Let D be a line distinct from A and B passing through $A\cap B$. By Lemma 3.1(ii),

$$\sum_{x\in A}\delta(x)-\sum_{y\in B}\delta(y)=(|A|-|B|)\,(|A|+\ell)\,. \tag{2}$$

Equalities (1) and (2) together give

$$\delta(A\cap C)=\delta(B\cap C). \tag{3}$$

Since $|A|\neq|B|$, Lemma 3.1(ii) implies that every point x of the line A is on at least $\delta(A\cap C)-1$ lines of size $|A|$. So, if v is the number of all points of $\mathscr{S}$, we have

$$v-1=(\delta(A\cap C)-1)\,(|A|-1)+(t-1)\,, \tag{4}$$

with t either equal to cardinality of the "last" line through x (which can, for the moment, be equal to or different from $|A|$).

By Lemma 3.1(ii), all lines distinct from B and passing through $B\cap C$ have the same size $|A|$. Thus,

$$v-1=(\delta(B\cap C)-1)\,(|A|-1)+(|B|-1)\,. \tag{5}$$

Comparing Equalities (4) and (5), and taking Equality (3) into account, we see that $t=|B|$. Hence there is a line E through x with $|E|=|B|$. Varying x, Equality (4) shows that every point outside the line B is on a unique line of size $|B|$, and hence all other lines have the same size $|A|$. If two lines of size $|B|$ intersect in a point r, then by Lemma 3.1(ii), all lines through r have size $|B|$ and hence $r\in B$ and the lines through r are the only lines in $\mathscr{S}$ of size $|B|$. Therefore, either the lines of size $|B|$ are pairwise disjoint or they all meet in a common point r.

Lemma 10. Let $\mathscr{S}$ be a $(2_i,0,\ell)$-interlaced linear space, with $\delta(x)\geq 4$ for each point x of $\mathscr{S}$ and let A and B be two lines with distinct size passing through a point p. If the lines of size $|B|$ are either pairwise disjoint or concurrent, then $\ell=0$.

Proof Case 1. Suppose that the lines of size $|B|$ are pairwise disjoint. Then by Lemma 3.1 (i), all points of $\mathscr{S}$ have the same degree n. Let D be a a line distinct from A and B passing through p. By Lemma 3.1(ii), we obtain $n(|A|-|B|)=(|A|-|B|)\,(|A|+\ell)$. Since $|A|\neq|B|$, we get $n=|A|+\ell$ which means that for any point x outside the line A, every line passing through x and disjoint from A is also disjoint from every line intersecting A and not passing through x. Therefore, every line passing through x intersects the line A. Thus $t=0$.

Case 2. Suppose that the lines of size $|B|$ are concurrent and all contain the point q. Note that all points except for q have the same degree n. Then by Lemma 3.1(i), applied to the triple (A,B,C), where the line $C\neq B$ intersects the line A and has size $|B|$, yields

$$n(|A|-(|B|-1))-b(q)=(|A|-|B|)\,(|B|+\ell)+n-\delta(q),$$

and so, since $|A| \neq |B|$, we get $n = v(b) + t$. Similarly as in Case 1, this implies $\ell = 0$.

Hence we can now proof a big part of Theorem A.

Theorem 2. If $\mathscr{S}$ is a $(2_i, 0, \ell)$-interlaced linear space containing two lines of different size, then $\ell = 0$ and $\mathscr{S}$ is a semiaffine plane.

Proof. If a linear space contains two lines of different size, then it must contain two concurrent lines of different size. If all points have degree at least 4, then the theorem follows from Lemma 3.1 and Lemma 3.1. If there are points with degree 2 or 3, then the theorem follows from Lemma 3.1 and Lemma 3.1.

Consequently, we may henceforth assume that $\mathscr{S}$ is a Steiner system (so all lines have the same size and hence all points have the same degree). As the statement of Theorem A shows, we have not been able to completely classify this case, but we will comment on the existence below.

Lemma 11. Let $\mathscr{S} = (\mathscr{P}, \mathscr{L}, \in)$ be a linear space with a points on every line, and with b lines through every point. Then

i. $(2_i, 0, \ell)$-interlacing implies both $(2_i, 1, 2(b-a-t)+1)$-interlacing and $(2_i, 2, 2a-b+t)$-interlacing;
ii. $(2_i, 1, \ell)$-interlacing implies both $(2_i, 0, b-a-\frac{1}{2}(\ell-1))$-interlacing and $(2_i, 2, a-\frac{1}{2}(ell+1))$-interlacing;
iii. $(2_i, 2, \ell)$-interlacing implies both $(2_i, 0, b+\ell-2a+1)$-interlacing and $(2_i, 1, 2(a-\ell)-1)$-interlacing.

Proof. For any pair of distinct but non-disjoint lines L, M, and every point x not in $L \cup M$, we clearly have $a = i_2(x, L \cup M) + (i_1(x, L, M)) + 1$ and $b = a + i_0(x, L \cup M) + i_1(x, M, L)$. Together with $i_1(x, L \cup M) = 2i_1(x, L, M) + 1 = 2i_1(x, M, L) - 1$, we see that, if one of $i_f(x, L \cup M)$, $f \in \{0, 1, 2\}$, is constant, then so are the others, and the above relations allow to compute these constants is one is given. We leave the easy actual computations to the reader.

This lemma implies that we may now consider $(2_i, 2, k)$-interlaced linear spaces with constant line size. We use the following standard notation: there are $s+1$ points on each line, and $t+1$ lines through each point.

If $s = 1$, then we have a complete graph on $t+2$ points, hence we may assume that $s > 1$.

We first construct a partial geometry.

Lemma 12. Let $\mathscr{S}$ be a $(2_i, 2, k)$-interlaced $(2, s+1, st+s+1)$ Steiner system with $s > 1$. Let L be any line. Then the points off L together with the lines intersecting L in precisely one point form a partial geometry with parameters $(s-1, s, k)$.

Proof. Clearly every line contains s points, and through every point there are $s+1$ lines (join that point with every point on L in $\mathscr{S}$). Now let M be a line of $\mathscr{S}$ meeting L in a point, and let x be a point not in $L \cup M$. Then, by assumption, there are k lines of $\mathscr{S}$ meeting both L and M. Hence the lemma.

Now we construct a symmetric 2-design.

Lemma 13. Let $\mathscr{S}$ be a $(2_i,2,k)$-interlaced $(2,s+1,st+s+1)$ Steiner system with $s>1$. Let p and q be two distinct points. Consider the following point-block structure $\mathscr{D}$. The points of $\mathscr{D}$ are the lines of $\mathscr{S}$ through p not containing q. Every block of $\mathscr{D}$ is associated to a line L of $\mathscr{S}$ through q and not containing p, and it contains the lines through p that meet L nontrivially (but not in p). Then $\mathscr{D}$ is a symmetric $2-(t,s,k)$ design.

Proof Since every line $L \neq pq$ through q has $1+s$ points, it is clear that all blocks have s points. Also, clearly the number of points of $\mathscr{D}$ is t, and this is also the number of blocks. Finally, given two different points A,B, there are $i_2(q,A\cup B)=k$ blocks containing them both. Hence we obtain a symmetric $2-=(t,s,k)$-design.

Lemma 13. Let $\mathscr{S}$ be a $(2_i,2,k)$-interlaced $(2,s+1,st+s+1)$ Steiner system. Then $s+1$ divides $t(t+1)$. Also, $s(s-1)=k(t+1)$.

Proof The first assertion follows from double counting the pairs (p,L), with p a point on the line L, which results in the number of lines of $\mathscr{S}$ being equal to $\frac{(st+s+1)(t+1)}{(s+1}$, which implies that $\frac{st(t+1}{s+1}$ is an integer. But since s and $s+1$ are relatively prime, the first assertion follows.

Now fix two intersecting lines L,M and count in two ways the number of pairs (p,A), where p is a point off $L\cup M$, and A is a line meeting $L\cup M$ in two points. Then the second assertion follows.

To finish the proof of Theorem A, we have to prove the mentioned relations between the parameters s,t,k. But these follow immediately from Lemma 3.1 and Lemma 3.1 and the known restrictions on the parameters of partial geometries (see [1] and symmetric designs (see [2] and [3]).

One could wonder whether there are any parameter sets surviving the tests of Theorem A. The answer is positive: for instance $(t,s,\ell)=(56,11,2)$ passes all tests. In fact, there is a symmetric 2-design with these parameters, but no corresponding partial geometry is known to exist. in fact, no parameter set of any known partial geometry passes all tests, but all our attempts to prove that this holds for all partial geometries failed, up to now.

4 Remarks

We remark that the definition of interlaced spaces can be easily generalized to all kinds of geometries, for instance to partial linear spaces. For instance, the $(1,1,1)$-interlaced partial linear spaces are precisely the (possibly degenerate) generalized quadrangles; the $(1,1,\alpha)$-interlaces partial linear spaces with either constant line size or constant point degree, are the partial geometries with parameters (s,t,α). As such, interlacing seems to be a natural condition and worthwhile of studying.

We end with some immediate properties of linear spaces using the terminology of interlacing.

Lemma 3.1 states equivalence between $(2_i, k, \ell)$-interlaced linear spaces, for $k = 0, 1, 2$, if the linear space is a Steiner system. The latter condition can also be stated with our notation.

Propasition 1. If $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ is a non-degenerate linear space and for any two intersecting lines L, M and any point x outside $L \cup M$ we have $i_1(x, L, M) = i_1(x, M, L) = t$, for some nonnegative integer then $\mathscr{S}$ is a Steiner system, and $(2, 1, 2t+1)$-interlaced.

Proof. Under the given conditions, we clearly have

$$i_1(x, L \cup M) = i_1(x, M, L) + i_1(x, L, M) + 1 = 2t + 1.$$

Therefore, $\mathscr{S}$ is a $(2, 1, 2t+1)$-interlaced space. Also, since

$$\begin{aligned} \delta(x) = & \quad |L| + i_1(x, M, L) + i_0(x, L \cup M), \\ = & |M| + i_1(x, L, M) + i_0(x, L \cup M) \text{ and} \\ = & \quad i_1(x, L, M) = i_1(x, M, L) = t, \end{aligned}$$

$\mathscr{S}$ is a Steiner system.

Finally, we remark that a $(1, 1, \ell)$-interlaced linear space always has a constant number of points on the lines.

5 Conclusion

A $(2_d, 2, k)$-*interlaced* is a linear space having property: for every pair L, L' of non-intersecting lines, and every point p not on either of these, there are precisely $|L| = |L'|$ (and hence a constant number k of) lines through p intersecting both L and L'. In a fortcoming paper of the same title, we will show that this property (with arbitrary constant k, not necessarily requiring $k = |L| = |L'|$) characterizes semiaffine planes together with projective spaces of dimension 3, and a small example obtained from the projective plane of order 2 by replacing a line by three lines of size 2. We call the latter linear space the *crooked Fano geometry*.

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