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Solution of Dirichlet problem for a square region in terms of elliptic functions

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Abstract: A broad class of steady-state physical problems can be reduced to finding the harmonic functions that satisfy certain boundary conditions. The Dirichlet problem for the Laplace (and Poisson) equations is one of the these mentioned problems. In this study, Dirichlet problem for the Laplace (also Poisson) differential equation in a square domain is expressed in terms of elliptic functions and the solution of the problem is based on the Green function and therefore on elliptic functions. To do this, it is made use of the basic consepts associated with elliptic integrals, conform mappings and Green functions.

Keywords: Dirichlet problem, Elliptic functions, Elliptic integral, Green function.

1 Introduction

Laplace's equation is one of the most significant equations in physics. It is the solution to problems in a wide variety of fields including thermodynamics and electrodynamics. Also, a broad class of steady-state physical problems can be reduced to finding the harmonic functions that satisfy certain boundary conditions. The Dirichlet problem for the Laplace (and Poisson) equation is one of the above-mentioned problems.

The Dirichlet problem for the Laplace equation is to find a function $U(z)$that is harmonic in a bounded domain $D\subset R^2$, is continuous up to the boundary $\partial D$of D, assumes the specified values $U_0(z)$ on the boundary ∂D, where $U_0(z)$ is a continuous function on ∂D, and can be formulated as

$$\nabla^2 U=0\,,\quad z\in D\,,\quad U\big|_{z\in\partial D}=U_0(z) \tag{1}$$

Here, for a point (x,y) in the plane R^2,one takes the complex notation $z=x+iy$, and$U(z)=U(x,y)$ and $U_0(z)=U_0(x,y)$ are real functions and $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator. Similary the Dirichlet problem for the Poisson equation can be formulated as

$$\nabla^2 U=h(z)\,,\quad z\in D\,,\quad U\big|_{z\in\partial D}=U_0(z) \tag{2}$$

In previous studies, the relation between the Dirichlet problem and the Cauchy problem was investigated [1]. Sezer developed a new method for the solution of the Dirichlet problem [2]. Lanzara [3] studied the Dirichlet problem for second degree elliptic linear equations with limited and measurable coefficients. The dependence on the variation of problem data of the solution of two-dimensional Dirichlet boundary-value problem for simply connected regions was

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also investigated [4].

In a previous study [1], the relation between Dirichlet problem and Couchy problem was investigated. Sezer [2] developed a new method for the solution of Dirichlet problem. Lanzara [3] studied Dirichlet problem for second degree elliptic linear equations with limited and measurable coefficients. The dependence on the variation of problem dat of the solution of two-dimensional Dirichlet boundary value problem for simply connected regions was investigated [4].Han and Hasebe [5] derived Green's funcion for a thermomechanical mixed boundary value problem problem of an infinite plane with an elliptic hole a pair of heat source and sink. Han and Hasebe [6] also reviwed Green's functions for a point heat source in various thermoelastic boundary value problems for an infinite plane with an inhomogeneity. Green function of the Dirichlet problem for the laplace differential equation in a rectangular domain was expressed in terms of elliptic functions and the solution of the problem was based on the Green function and therefore on elliptic functions by Kurt ,Sezer and Çelik et al [7].Hsiao et al. [8] showed an equivalence between the weak solition and the various boundary integral solutions, and described a coupling procedure for an exterior initial boundary value problem for the nonhomogeneous heat equation.The problem of the one-dimensional heat equatιon with nonlinear boundary conditions was studied by Tao [9]. Hansen [10] studied a boundary integral method for the solution of the heat equation in an unbounded domain D in R^2. The application of spectral methods for solving the one-dimensional heat equation was presented by Saldana et al. [11]. Al-Najem et al. [12] estimated the surface temparature in two-dimensional steady-state in a rectangulear region by two different methods, the singular value decomposition with boundary element method and the least-squares approach with integral transform method. The Green function of the Dirichlet problem for the Laplace diferential equation in a triangle region was expessed in terms of elliptic functions and the solution of problem was based on the Green function, and therefore on elliptic functions by Kurt and Sezer [13]. Green function of the two-dimensional heat equation in a square region was expresed in terms of elliptic functions and the solution of the problem was based on the Green function and therefore on elliptic functions by Kurt et al [14]. Least Square Method (LSM), Collocation Method (CM) and new approach which called Akbari-Ganji's Method (AGM) are applied to solve the nonlinear heat transfer equation of fin with power-law temperature-dependent both thermal conductivity and heat transfer coefficient by S.T. Ledari et al [15].

As it is knows, the solution of the Dirichlet problem by the method of separation of variables may be obtained only for a restricted class od domains D with a sufficiently simple boundary ∂D. The conformal mappings yield a sufficiently universal algorithm for the solution of the Dirichlet problem for two-dimensional domains. These permit constructing a Green function of the Dirichlet problem for the Laplace (and Poisson) equation in a Dconformally onto the unit circle or upper half-plane, cannot ne obtained in terms of elliptic functions.

Our purpose in this paper, first, is to determine the analytic function. Which maps the square domain D onto the upper half-plane or the unit circle in terms of elliptic functions using the Schwarz-Christoffel transformation and conformal mapping, and then, to find the solution of the Dirichlet problem for the square domain in terms of elliptic functions, by means of the relation between the obtained analytic function and the Green function.

2 Elliptic integrals and functions

The integral

$$\int_0^t \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = \int_0^u du_1 = u = sn^{-1}(t,k) = F(\phi,k) \;\; t = \sin\phi \tag{3}$$

is called the normal elliptic integral of the first kind, where k, $(0 < k < 1)$is any number. When $t = 1$, Eq. (3) is said to be complete and becomes

$$\int_0^t \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = \int_0^K du_1 = F\left(\frac{\pi}{2}, k\right) \equiv K(k) \equiv K \tag{4}$$

or

$$\int_0^t \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = \int_0^K du_1 = F\left(\frac{\pi}{2}, k'\right) \equiv K'(k) \equiv K(k') \equiv K' \tag{5}$$

where, the number k is the modulus and k', $(0 < k' < 1)$ is the complementary modulus, such that $k'^2 = 1 - k^2$. If $k = 0$ in Eq. (3), one finds that $u = \sin^{-1} t$ or $t = \sin u$. When $k \neq 0$,the integral (3) is denoted by $u = sn^{-1}(t,k)$ or briefly $u = sn^{-1}t$ or $t = sn\,u$. The function $sn\,u$ is called Jacobian elliptic function. Two other Jacobian elliptic functions can be defined by $cn\,(u,k) = (1-k^2)^{1/2}$.

3 The conform mapping of a square domain and determination of green function

The function $w = F(z) = \frac{snz}{1+cnz}$ maps the rectangle $-K < Re(z) < K, \quad -K' < Im(z) < K'$ onto the unit circle $|w| < 1$. If $k^2 = 0.5$ then $K = K'$ and the conformal mapping of the square in the z-plane onto the unit circle $|w| < 1$ in the w-plane can be written as

$$w = F(z) = \frac{snz}{1+cnz} \quad -K < Re(z) < K, \quad -K < Im(z) < K \tag{6}$$

Green function $G(z,\zeta)$ of the Dirichlet problem for the Laplace equation in the domain D is defined by

$$G(z,\zeta) = \frac{1}{2\pi}\ln|z-\zeta| + g(z,\zeta) \quad z \in D\,,\ \zeta \in D \tag{7}$$

where g is a harmonic function in D for each $\zeta \in D$ and $g(z,\zeta) = -1/(2\pi)\ln|z-\zeta|$ then $G(z,\zeta) = 0$, for each $z \in \partial D$, $z = x+iy$ and $\zeta = \xi + i\eta$.

When the domain $D$is simply connected, the determination of the mentioned Green function can be reduced to the problem of determining an analytic function which specifies a mapping of D onto the upper half-plane Im$W > 0$ or the unit circle $|\omega| < 1$.This is so because, $W = F(z)$ is an analytic function which maps the domain D in the z-plane onto the upper half-plane of the W-plane, with $F'(z) \neq 0$ in D then the mapping is one-to-one.

$$G = \frac{1}{2}\ln\left|\frac{F(z)-F(\zeta)}{F(z)-\overline{F(\zeta)}}\right|, \quad z = x+iy\,,\ \zeta = \xi + i\eta \tag{8}$$

and, if the analytic function $W = F(z)$ maps D onto the unit circle $|\omega| < 1$, then the Green function of the Dirichlet problem for the Laplace operator in D becomes

$$G(z,\zeta) = \frac{1}{2\pi}\ln|\omega(z-\zeta)|\ ,\ W(z,\zeta) = \frac{F(z)-F(\zeta)}{1-F(z)\overline{F(\zeta)}}. \tag{9}$$

Consequently, if one takes D as the square $A_1(K,-K)$,$A_2(K,K)$, $A_3(-K,K)$, $A_4(-K,-K)$ then from Eqs. (6) and (9), the Green function for the square is found in the form

$$G=\frac{1}{2\pi}\ln\left|\frac{\frac{snz}{1+cnz}-\frac{sn\zeta}{1+cn\zeta}}{1-\frac{snz}{1+cnz}\overline{\frac{sn\zeta}{1+cn\zeta}}}\right|. \tag{10}$$

4 The solution of the Dirichlet problem

The solution of the Dirichlet problem for the Poisson equation (2) in D can be obtained as

$$U(z)=\int_D\int G(z,\zeta)h(\zeta)d\xi\,d\eta+\int_{\partial D}\frac{G(z,\zeta)}{\partial n}U_0(\zeta)\,|d\zeta| \tag{11}$$

where G is the Green function for the domain D and $\partial/\partial n$ denotes differentiation along an outward normal to the boundary ∂D of D with respect to ζ. Taking the domain D as the square $A_1(K,-K)$,$A_2(K,K)$,$A_3(-K,K)$,$A_4(-K,-K)$ and the boundary ∂D of D as the circumference $\partial D=\overline{A_4A_1}\cup\overline{A_1A_2}\cup\overline{A_2A_3}\cup\overline{A_3A_4}$, one may write the conditions

1. $\eta=-K,\quad d\eta=0,\quad -K\le\xi\le K$ on $\overline{A_4A_1}$
2. $\xi=K,\quad d\xi=0,\quad -K\le\eta\le K$ on $\overline{A_1A_2}$
3. $\eta=K,\quad d\eta=0,\quad -K\le\xi\le K$ on $\overline{A_2A_3}$
4. $\xi=-K,\quad d\xi=0,\quad -K\le\eta\le K$ on $\overline{A_3A_4}$.

Thus, from Eq. (11), the solution of Eq. (2) in the above square becomes

$$U(z)=\int_{-K}^{K}\int_{-K}^{K}G(z,\zeta)h(\zeta)d\xi\,d\eta-\int_{-K}^{K}\left[G_\xi^2(z,\zeta)+G_\eta^2(z,\zeta)\right]^{1/2}U_0(\zeta)|_{\eta=-K}^{K}\,d\xi$$
$$+\int_{-K}^{K}\left[G_\xi^2(z,\zeta)+G_\eta^2(z,\zeta)\right]^{1/2}U_0(\zeta)|_{\xi=-K}^{K}\,d\eta. \tag{12}$$

In the case of $h(z)=0$, the solution of the Dirichlet problem for the Laplace differential equation (1) in the above square is obtained in terms of elliptic functions as :

$$U(z)=\int_{-K}^{K}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0(\zeta)|_{\xi=-K}^{K}\,d\eta-\int_{-K}^{K}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0(\zeta)|_{\eta=-K}^{K}\,d\xi \tag{13}$$

where the Green function G is defined by

$$G=\frac{1}{2\pi}Re\ln\left[\frac{\frac{snz}{1+cnz}-\frac{sn\zeta}{1+cn\zeta}}{1-\frac{snz}{1+cnz}\overline{\frac{sn\zeta}{1+cn\zeta}}}\right],\quad z=x+iy,\ \zeta=\xi+i\eta \tag{14}$$

according to Eq. (10). The boundary values $K(k)$ is the complete elliptic integrals and is tabulated for $k^2=0.5$.

5 Discussion

The method of conformal mapping is a more flexible and far-reaching tool fort he Laplace equation in the plane. In a certain sense conformal mapping provides the analogue for elliptic differential equations of the method of characteristics developed for hyperbolic differential equations [16]. However like characteristic coordinates, it is only applicable in the case of two independent variables.

Following the way in the present paper, the Dirichlet problem for the Laplace (also Poisson) differential equation in similar regions such as the exterior of rectangle and ellipse, and square can be solved in terms of elliptic functions; thus, contribution may be provided for the solution of similar problems in physics and engineering.

The most important advantage of present method is that the result is obtained in terms of elliptic functions; because expressing the result in terms of elliptic functions facilitates many physics and engineering problems. On the other hand, the disadvantage is that it is rather difficult to find the derivatives and integrals of the elliptic functions and the Green function necessary for the solution of the Dirichlet problemin the required domain.

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