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Chelyshkov collocation approach to solve the systems of linear functional differential equations

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Received: 12 June 2015, Revised: 18 June 2015, Accepted: 6 November 2015
Published online: 26 November 2015

Abstract: In this paper, we present a new collocation method based on Chelyshkov polynomials for solving the system of functional differential equations under the initial-boundary conditions.By means of Chelyshkov polynomials and collocation points, this method converts the so-called system into a matrix equation, which involves the unknown Chelyshkov coeffcients. We give some illustrative examples, which arise in physics, biology, chemistry and mechanics and so on, to indicate the reliability and efficiency of the method. Also, a technique based on residual functions is performed to check the accuracy of the problem.

Keywords: Systems of delay differential-difference equations; Chelyshkov polynomials; Numerical solutions; Residual functions.

1 Introduction

Mathematical models, especially those related to systems of delay differential and differential-difference equations, are of the great importance in the real-life such as various mechanics, physics, biology, economy, epidemics, population dynamic models, automatic control systems, neural networks, chaotic systems and so on [1-10] (For more details, see the references therein). Therefore, during the last few decades, a number of mathematical methods that are aimed at solving the so-called systems have appeared in the research literature such as variational iteration method [11], Differential transformation method [12], Haar functions method [13], homotopy analysis method [14], homotopy perturbation method [15] and Tau method [16].

In addition to these methods mentioned in the literature, systems of linear differential, integral, integro-differential and differential-difference equations were solved using the collocation methods based on Sezer's matrix methods, which are derived from special functions as Taylor, Chebyshev, Legendre, Laguerre, Hermite, Bessel and so on [17-25].

The area of orthogonal polynomials is a very active research area in mathematics as well as in applications in mathematical physics, engineering and computer science. One of the latest set of orthogonal polynomials is the set of the Chelyshkov polynomials $\{C_{N0}(t), C_{N1}(t), ..., C_{NN}(t), ...\}$. Recently, these polynomials have created by Chelyshkov in [26-32], which are orthogonal over the interval [0,1] with respect to the weight function $w(x) = 1$, and are explicitly defined by

$$C_{Nn}(t) = \sum_{j=0}^{N-n} (-1)^j \binom{N-n}{j} \binom{N+n+j+1}{N-n} t^{n+j}, n = 0, 1 \ldots N. \tag{1}$$

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This yields The Rodrigues' type representation

$$C_{Nn}(t) = \frac{1}{(N-n)!}\frac{1}{t^{n+1}}\frac{d^{N-n}}{dt^{N-n}}\left(t^{N+n+1}(1-t)^{N-n}\right), n = 0,1,\ldots,N, \tag{2}$$

and the following orthogonality relations

$$\int_0^1 C_{Np}(t)C_{Nq}(t)dx = \begin{cases} 0, \ p \neq q \\ \frac{1}{p+q+1}, \ p = q \end{cases}, p,q,...,N. \tag{3}$$

Also it follows from this relation that
$$\int_0^1 C_{Nn}(t)dx = \int_0^1 t^n dx = \frac{1}{n+1}.$$

By using The Rodrigues' formula and the Cauchy integral formula for derivatives of an analytic function, one can obtain the integral relation

$$C_{Nn}(t) = \frac{1}{2\pi i}\frac{1}{t^{n+2}}\int\limits_{\Omega_1}\frac{\overline{z}^{(N+2+n)}(1-z)^{N-n}}{(z-t^{-1})^{N-n+1}}dz,$$

where Ω_1 is a closed curve, which encloses the point $z = t^{-1}$.

Chelyshkov polynomials $C_{Nn}(t)$ have the analogous properties to those of the classical orthogonal polynomials. In fact, these polynomials are an example of such alternative orthogonal ones, which are not solutions of the hypergeometric type, but can be expressed in terms of the Jacobi ones. In addition, they can also be connected to hypergeometric functions, orthogonal expontional polynomials, and the Jacobi polynomials $P_k^{(\alpha,\beta)}$ by the following relation,

$$C_{Nn}(t) = t^n P_{N-n}^{(2n,1)}(1-2t), n = 0,1\ldots N.$$

Hence, they keep distinctively attributes of the classical orthogonal polynomials and may be facilitated to different problems on approximation. In the family of orthogonal polynomials $\{C_{Nn}(t)\}$, every member has degree N with N-n simple roots. Hence, for every N if the roots of the polynomial are chosen as node points, then an acurate numerical quadrature can be derived.

In this study, we consider the system of functional differential equations with variable coefficients in the form,

$$\sum_{r=0}^{m}\sum_{i=1}^{k}\sum_{s=0}^{S}\left\{P_{ji}^{r,s}(t)y_i^{(r)}(\lambda t+\mu_s)+Q_{ji}^{r}(t)y_i^{(r)}(t)\right\} = g_j(t), \text{j} = 1,2,...,\text{k}, \ 0 \le t \le 1 \tag{4}$$

subject to the initial-boundary conditions

$$\sum_{i=0}^{m-1}\left(a_{ji}^{n}y_n^{(i)}(0)+b_{ji}^{n}y_n^{(i)}(1)\right) = \alpha_{ni}, \quad j = 0,1,\cdots,.m-1, \quad n = 1,2,\cdots,k \tag{5}$$

where a_{ji}^n, b_{ji}^n, λ, μ *and* $\alpha_{ni,}$ are real or complex constants. Meanwhile $P_{ji}^r(t)$ *and* $Q_{ji}^r(t)$ are continuous functions defined in $0 \le t \le 1$. Our aim in this paper is to find an approximate solutions of Eq. (4) under the initial-boundary conditions (5) in the truncated Chelyshkov series, based on (1) or (2), form

$$y_i(t) = \sum_{n=0}^{N} a_{i,n} C_{N,n}(t), \quad i = 1,\cdots,k, \;\; 0 \le t \le 1 \tag{6}$$

so that $a_{i,n}$, n = 0,1,2,...,N are the unknown Chelyshkov coefficients. Here, N is chosen any positive integer such that $N \ge k,m$.

2 Fundamental matrix relations

First, we can write $y_i(t)$ and their derivatives in the matrix forms as follows:

$$y_i(t) = \mathbf{C}(t)\mathbf{A_i} \;\; or \;\; y_i(t) = \mathbf{T}(t)\mathbf{C}\mathbf{A_i}, \;\; i = 1,\ldots,k \tag{7}$$

$$\begin{aligned}
y_i^{(1)}(t) &= \mathbf{C}^{(1)}(t)\mathbf{A_i} = \mathbf{T}(t)\mathbf{B}\mathbf{C}\mathbf{A_i} \\
y_i^{(2)}(t) &= \mathbf{C}^{(2)}(t)\mathbf{A_i} = \mathbf{T}(t)\mathbf{B}^2\mathbf{C}\mathbf{A_i} \\
&\;\;\vdots \\
y_i^{(r)}(t) &= \mathbf{C^{(r)}(t)A_i} \;\; \mathbf{or} \;\; \mathbf{y_i^{(r)}(t) = T(t)B^rCA_i}, \;\; \mathbf{r = 1,\ldots,m}
\end{aligned} \tag{8}$$

where

$$\mathbf{C}(t) = \begin{bmatrix} C_{N0} \; C_{N1} \; \ldots \; C_{NN} \end{bmatrix}, \quad \mathbf{T}(t) = \begin{bmatrix} 1 \; t \; \ldots \; t^N \end{bmatrix}$$

if N is odd, from (1) and (3)

$$\mathbf{C} = \begin{bmatrix}
\binom{N}{0}\binom{N+1}{N} & 0 & \ldots & 0 & 0 \\
-\binom{N}{1}\binom{N+2}{N} & \binom{N-1}{0}\binom{N+2}{N-1} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\binom{N}{N-1}\binom{2N}{N} & -\binom{N-1}{N-2}\binom{2N}{N-1} & \ldots & \binom{1}{0}\binom{2N}{1} & 0 \\
-\binom{N}{N}\binom{2N+1}{N} & \binom{N-1}{N-1}\binom{2N+1}{N-1} & \ldots & -\binom{1}{1}\binom{2N+1}{1} & 1
\end{bmatrix}_{(N+1)x(N+1)}$$

if N is even

$$\mathbf{C} = \begin{bmatrix}
\binom{N}{0}\binom{N+1}{N} & 0 & \ldots & 0 & 0 \\
-\binom{N}{1}\binom{N+2}{N} & \binom{N-1}{0}\binom{N+2}{N-1} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\binom{N}{N-1}\binom{2N}{N} & \binom{N-1}{N-2}\binom{2N}{N-1} & \ldots & \binom{1}{0}\binom{2N}{1} & 0 \\
\binom{N}{N}\binom{2N+1}{N} & -\binom{N-1}{N-1}\binom{2N+1}{N-1} & \ldots & -\binom{1}{1}\binom{2N+1}{1} & 1
\end{bmatrix}_{(N+1)x(N+1)}$$

$$\mathbf{B}=\begin{bmatrix} 0\ 1\ 0\ \dots\ 0 \\ 0\ 0\ 2\ \dots\ 0 \\ 0\ 0\ 0\ \ddots\ 0 \\ 0\ 0\ 0\ \dots\ N \\ 0\ 0\ 0\ \dots\ 0 \end{bmatrix}_{(N+1)x(N+1)}, \quad \mathbf{A_i}=\begin{bmatrix} a_{i0}\ a_{i1}\ \dots\ a_{iN}\end{bmatrix}^T, i=1,...,k.$$

Replacing $(\lambda t+\mu_s)$ by t in the relation (8) we have the matrix form

$$y_i^{(r)}(\lambda t+\mu_s)=\mathbf{C}^{(r)}(\lambda t+\mu_s)\mathbf{A_i}=\mathbf{T}(\lambda t+\mu_s)\mathbf{B}^r\mathbf{A_i},\ r=1,...,m. \tag{9}$$

The relation between the matrices $\mathbf{T}(\lambda t+\mu_s)$ and $\mathbf{T}(t)$ is

$$\mathbf{T}(\lambda t+\mu_s)=\mathbf{T}(t)\mathbf{M}(\lambda,\mu_s) \tag{10}$$

such that, for $\lambda\neq 0$ and $\mu_s\neq 0$ [33]

$$\mathbf{M}(\lambda,\mu_s)=\begin{bmatrix} \binom{0}{0}\lambda^0\mu_s^0 & \binom{1}{0}\lambda^0\mu_s^1 & \binom{2}{0}\lambda^0\mu_s^2 & \dots & \binom{N}{0}\lambda^0\mu_s^N \\ 0 & \binom{1}{1}\lambda^1\mu_s^0 & \binom{2}{1}\lambda^1\mu_s^1 & \dots & \binom{N}{1}\lambda^1\mu_s^N \\ 0 & 0 & \binom{2}{2}\lambda^2\mu_s^0 & \dots & \binom{N}{2}\lambda^2\mu_s^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \binom{N}{N}\lambda^N\mu_s^N \end{bmatrix}$$

and for $\lambda\neq 0$ and $\mu_s=0$

$$\mathbf{M}(\lambda,0)=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda^N \end{bmatrix}$$

We have the following matrix relation by substituting Eq.(10) into Eq.(9)

$$y_i^{(r)}(\lambda t+\mu_s)=\mathbf{T}(t)\mathbf{M}(\lambda,\mu_s)\mathbf{B}^r\mathbf{A_i},\ r=1,\dots,m \text{ and } i=1,\dots,k. \tag{11}$$

By using the relations (8),(10) and (11),we find the following matrix forms

$$\mathbf{y}^{(r)}(t)=\overline{\mathbf{T}}(t)\overline{\mathbf{B}}^r\overline{\mathbf{C}}\mathbf{A},\ r=0,1,...,m \tag{12}$$

and

$$\mathbf{y}^{(r)}(\lambda t+\mu_s)=\overline{\mathbf{T}}(t)\overline{\mathbf{M}}(\lambda,\mu_s)\overline{\mathbf{B}}^r\overline{\mathbf{C}}\mathbf{A}, r=0,1,...,m \tag{13}$$

where

$$\mathbf{y}^{(r)}(t)=\begin{bmatrix} y_1^{(r)}(t)\\ y_2^{(r)}(t)\\ \vdots\\ y_k^{(r)}(t)\end{bmatrix},\mathbf{y}^{(r)}(\lambda t+\mu_s)=\begin{bmatrix} y_1^{(r)}(\lambda t+\mu_s)\\ y_2^{(r)}(\lambda t+\mu_s)\\ \vdots\\ y_k^{(r)}(\lambda t+\mu_s)\end{bmatrix},\mathbf{A}=\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\\ \vdots\\ \mathbf{A}_k\end{bmatrix},\overline{\mathbf{T}}(t)=\begin{bmatrix}\mathbf{T}(t) & 0 & \dots & 0\\ 0 & \mathbf{T}(t) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{T}(t)\end{bmatrix},$$

$$\overline{\mathbf{M}}(\lambda,\mu_s)=\begin{bmatrix}\mathbf{M}(\lambda,\mu_s) & 0 & \dots & 0\\ 0 & \mathbf{M}(\lambda,\mu_s) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{M}(\lambda,\mu_s)\end{bmatrix},\overline{\mathbf{B}}^r=\begin{bmatrix}\mathbf{B}^r & 0 & \dots & 0\\ 0 & \mathbf{B}^r & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{B}^r\end{bmatrix} \text{ and } \overline{\mathbf{C}}=\begin{bmatrix}\mathbf{C} & 0 & \dots & 0\\ 0 & \mathbf{C} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{C}\end{bmatrix}.$$

3 Method of solution

In this section, we convert the system (4) to linear systems of matrix equations which can be easily solved. Firstly, by means of the matrix relations (12) and (13), we can write the system (4) in the matrix form

$$\sum_{r=0}^{m}\sum_{s=0}^{S}\left\{\mathbf{P}_{\mathbf{r},\mathbf{s}}(t)\mathbf{y}^{(r)}(\lambda t+\mu_s)+\mathbf{Q}_{\mathbf{r}}(t)\mathbf{y}^{(r)}(t)\right\}=\mathbf{g}(t), \tag{14}$$

where

$$\mathbf{P}_{\mathbf{r},\mathbf{s}}(t)=\begin{bmatrix}P_{11}^{r,s}(t) & P_{12}^{r,s}(t) & \dots & P_{1k}^{r,s}(t)\\ P_{21}^{r,s}(t) & P_{22}^{r,s}(t) & \dots & P_{2k}^{r,s}(t)\\ \vdots & \vdots & \ddots & \vdots\\ P_{k1}^{r,s}(t) & P_{k2}^{r,s}(t) & \dots & P_{kk}^{r,s}(t)\end{bmatrix},\mathbf{Q}_{\mathbf{r}}(t)=\begin{bmatrix}Q_{11}^{r}(t) & Q_{12}^{r}(t) & \dots & Q_{1k}^{r}(t)\\ Q_{21}^{r}(t) & Q_{22}^{r}(t) & \dots & Q_{2k}^{r}(t)\\ \vdots & \vdots & \ddots & \vdots\\ Q_{k1}^{r}(t) & Q_{k2}^{r}(t) & \dots & Q_{kk}^{r}(t)\end{bmatrix},\mathbf{g}(\mathrm{t})=\begin{bmatrix}g_1(t)\\ g_2(t)\\ \vdots\\ g_k(t)\end{bmatrix}$$

By substituting the collocation points defined by

$$t_q=\frac{1}{N}q,\ q=0,1,...,N$$

into Eq. (14), the system of the matrix equations is obtained as

$$\sum_{r=0}^{m}\sum_{s=0}^{S}\left\{\mathbf{P}_{\mathbf{r},\mathbf{s}}(t_q)\mathbf{y}^{(r)}(\lambda t_q+\mu_s)+\mathbf{Q}_{\mathbf{r}}(t_q)y^{(r)}(t_q)\right\}=\mathbf{g}(t_q),$$

or briefly expressed as follows

$$\sum_{r=0}^{m}\sum_{s=0}^{S}\left\{\mathbf{P}_{\mathbf{rs}}\overline{\mathbf{Y}}^{(r)}+\mathbf{Q}_{\mathbf{r}}\mathbf{Y}^{(r)}\right\}=\mathbf{G}, \tag{15}$$

where

$$\mathbf{P}_{\mathbf{rs}}=\begin{bmatrix}\mathbf{P}_{\mathbf{r},\mathbf{s}}(t_0) & 0 & \dots & 0\\ 0 & \mathbf{P}_{\mathbf{r},\mathbf{s}}(t_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{P}_{\mathbf{r},\mathbf{s}}(t_N)\end{bmatrix},\overline{\mathbf{Y}}^{(r)}=\begin{bmatrix}\mathbf{y}^{(r)}(\lambda t_0+\mu_i)\\ \mathbf{y}^{(r)}(\lambda t_1+\mu_i)\\ \vdots\\ \mathbf{y}^{(r)}(\lambda t_N+\mu_i)\end{bmatrix},\mathbf{Y}^{(r)}=\begin{bmatrix}\mathbf{y}^{(r)}(t_0)\\ \mathbf{y}^{(r)}(t_1)\\ \vdots\\ \mathbf{y}^{(r)}(t_N)\end{bmatrix},$$

$$\mathbf{Q_r}=\begin{bmatrix}\mathbf{Q_r}(t_0) & 0 & \dots & 0\\ 0 & \mathbf{Q_r}(t_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{Q_r}(t_N)\end{bmatrix},\mathbf{G}=\begin{bmatrix}\mathbf{g}(t_0)\\ \mathbf{g}(t_1)\\ \vdots\\ \mathbf{g}(t_N)\end{bmatrix}.$$

Using relations (12) and (13) in Eq.(15), we have the fundamental matrix equation

$$\sum_{r=0}^{m}\sum_{s=0}^{S}\left\{\mathbf{P_{rs}}\mathbf{T}\overline{\mathbf{M}}(\lambda,\mu_s)\overline{\mathbf{B}}^{r}\overline{\mathbf{C}}+\mathbf{Q_r}\mathbf{T}\overline{\mathbf{B}}^{r}\overline{\mathbf{C}}\right\}=\mathbf{G}, \tag{16}$$

where

$$\mathbf{T}=\left[\overline{\mathbf{T}}(t_0)\ \overline{\mathbf{T}}(t_1)\ \dots\ \overline{\mathbf{T}}(t_N)\right]^T.$$

Briefly, we can write Eq.(16) in the form

$$\mathbf{WA}=\mathbf{G}\quad[\mathbf{W};\mathbf{G}] \tag{17}$$

which corresponds to a system of the linear algebraic equations with the unknown Chelyshkov coefficients elements $a_{i,n}, i=1,2,...,k, n=0,1,...,N$. us find a matrix representation of the conditions given in (5). Using the relation (12), the matrix representation of the initial and boundary conditions which depend on the Chelyshkov coefficients matrix is obtained as

$$\sum_{j=0}^{m-1}\left\{\mathbf{a_j}\overline{\mathbf{T}}(0)+\mathbf{b_j}\overline{\mathbf{T}}(1)\right\}\overline{\mathbf{B}}^{j}\overline{\mathbf{C}}\mathbf{A}=\alpha, \tag{18}$$

where

$$\mathbf{a_j}=\begin{bmatrix}\mathbf{a_j}^1 & 0 & \dots & 0\\ 0 & \mathbf{a_j}^2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{a_j}^k\end{bmatrix},\mathbf{b_j}=\begin{bmatrix}\mathbf{b_j}^1 & 0 & \dots & 0\\ 0 & \mathbf{b_j}^2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{b_j}^k\end{bmatrix}\ \text{and}\ \begin{array}{l}\alpha=\left[\alpha_1\ \alpha_2\ \dots\ \alpha_k\right]^T\\ \mathbf{a_j}^i=\left[a^i_{0j}\ a^i_{1j}\ \cdots\ a^i_{m-1j}\right]^T\\ \mathbf{b_j}^i=\left[b^i_{0j}\ b^i_{1j}\ \dots\ b^i_{m-j}\right]^T\end{array}$$

Thus, the matrix form (18) for the conditions becomes

$$\mathbf{UA}=\alpha\ \text{ or }\ [\mathbf{U};\alpha];\mathbf{U}=\sum_{j=0}^{m-1}\left\{a_j\overline{\mathbf{T}}(0)+b_j\overline{\mathbf{T}}(1)\right\}\overline{\mathbf{B}}^{j}\overline{\mathbf{C}} \tag{19}$$

Lastly, by replacing the last rows of the augmented matrix (17) by the rows of matrix $[\mathbf{U};\alpha]$, we have the new augmented matrix

$$\widetilde{\mathbf{W}}\widetilde{\mathbf{A}}=\widetilde{\mathbf{G}}\text{ or }\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right] \tag{20}$$

which is a linear system of algebraic equations. The unknown Chelyshkov coefficients can be found by solving this system. When the unknown Chelyshkov coefficients $a_{i,0},\ a_{i,1},\dots,a_{i,N}$ are substituted in Eq. (6), we obtain the Chelyshkov polynomial solution

$$y_i(t)\cong\sum_{n=0}^{N}a_{i,n}C_{N,n}(t),\ \mathrm{i}=1,...,\mathrm{k},\ 0\leq\mathrm{t}\leq1$$

On the other hand, when $|\widetilde{\mathbf{W}}|=0$, if $\mathrm{rank}\widetilde{\mathbf{W}}=\mathrm{rank}\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]<\mathrm{k(N+1)}$, then we may find a particular solution. Otherwise there is not a solution. We can easily check the accuracy of this solution as follows:

Since the truncated Chelyshkov series (6) is approximate solution of (4), when the function $y_{i,N}(t), i=1,2,...,k$ and their derivatives are substituted in Eq.(4), the resulting equation must be satisfied approximately; that is, for $t_q \in [0,1], q = 0,1,....$

$$E_{j,N}(t_q) = \left| \sum_{r=0}^{m} \sum_{i=1}^{k} \sum_{s=0}^{S} \left\{ P_{ji}^{r,s}(t_q) y_i^{(r)}(\lambda t_q + \mu_s) + Q_{ji}^{r}(t_q) y_i^{(r)}(t_q) \right\} - g_j(t_q) \right| \cong 0, \mathrm{j} = 1,2,...,\mathrm{k} \tag{21}$$

or $E_{j,N}(t_q) \leq 10^{-r_p}$ (r_p is any positive number). If $\max(10^{-r_p}) = 10^{-r}$ (r is any positive integer) is prescribed, then the truncation limit N is increased until the differences $E_{j,N}(t_q)$ at each of the points become smaller than the prescribed 10^{-r}, see [34-38]. If when N is sufficiently large enough, then the error decreases. On the other hand, the error can be estimated by system,

$$E_{j,N}(t) = \sum_{r=0}^{m} \sum_{i=1}^{k} \sum_{s=0}^{S} \left\{ P_{ji}^{r,s}(t) y_i^{(r)}(\lambda t + \mu_s) + Q_{ji}^{r}(t) y_i^{(r)}(t) \right\} - g_j(t), \ j = 1, \ ,2,...,\mathrm{k}$$

4 Illustrations

In this section, some numerical examples on the problem (4) are given to illustrate the accuracy and effectiveness properties of the method.

Example 1. [23]. Let us consider the following linear system of second-order advanced differential-difference equations,

$$\begin{cases} y_1^{(2)}(t-1/2) + 2t y_1^{(1)}(t+1/3) - t y_2(t-1/2) + t^2 y_3(t-1) = g_1(t) \\ y_2^{(2)}(t-1/4) - t y_1^{(1)}(t+1/5) - t y_2(t-1/6) + 5 y_3(t-1/2) = g_2(t) \\ y_3^{(2)}(t+1/3) - t y_1^{(1)}(t-1/6) + y_3^{(1)}(t-1/3) + 3 y_1(t+1/4) + 2t y_2(t+1/3) = g_3(t) \end{cases} , 0 \leq t \leq 1 \tag{22}$$

and the initial conditions

$$y_1(0) = 0, \mathrm{y'}_1(0) = 1, \mathrm{y}_2(0) = 1, \mathrm{y'}_2(0) = 0, \mathrm{y}_3(0) = 1, \mathrm{y'}_3(0) = 1$$

where

$$\begin{cases} [l] g_1(t) = -\sin(t-1/2) + 2t\cos(t+1/3) - t\cos(t-1/2) + t^2 e^{t-1}, \\ g_2(t) = -\cos(t-1/4) - t\cos(t+1/5) + t\cos(t-1/6) + 5e^{t-1/2}, \\ g_3(t) = -t\cos(t-1/6) + te^{t-1/3} + 3\sin(t+1/4) + 2t\cos(t+1/3) + e^{t+1/3}, \end{cases}$$

and the exact solutions are $y_1(t) = \sin t, y_2(t) = \cos t$ and $y_3(t) = e^t$.

For $N = 3$, the approximate solutions by the truncated Chelyshkov series and the collocation points are, respectively, given by

$$y_i(t) = \sum_{n=0}^{3} a_n C_{3,n}(t), \mathrm{i} = 1,2,3$$

and $t_0 = 0, t_1 = 1/3, t_2 = 2/3, t_3 = 1$. The fundamental matrix equation of the problem is as follows,

$$\left\{ \begin{array}{l} \mathbf{P_{01}T\overline{M}_{1,-\frac{1}{2}}\overline{C} + P_{04}T\overline{M}_{1,-\frac{1}{6}}\overline{C} + P_{05}T\overline{M}_{1,-1}\overline{C} + P_{06}T\overline{M}_{1,\frac{1}{3}}\overline{C} + P_{07}T\overline{M}_{1,\frac{1}{4}}\overline{C} + P_{12}T\overline{M}_{1,-\frac{1}{3}}\overline{BC}} \\ \mathbf{+P_{14}T\overline{M}_{1,-\frac{1}{6}}\overline{BC} + P_{16}T\overline{M}_{1,\frac{1}{3}}\overline{BC} + P_{18}T\overline{M}_{1,\frac{1}{5}}\overline{BC} + P_{21}T\overline{M}_{1,-\frac{1}{2}}\overline{B}^2\overline{C} + P_{23}T\overline{M}_{1,-\frac{1}{4}}\overline{B}^2\overline{C} + P_{26}T\overline{M}_{1,\frac{1}{3}}\overline{B}^2\overline{C}} \end{array} \right\} \mathbf{A} = \mathbf{G},$$

where

$$\mathbf{P_{ij}} = \mathbf{diag}\left[\mathbf{P_{i,j}}(0),\, \mathbf{P_{i,j}}(1/3),\, \mathbf{P_{i,j}}(2/3),\, \mathbf{P_{i,j}}(1)\right], \quad i = 0,1,2 \text{ and } j = 1,2,\ldots,8$$

$$\mathbf{P_{0,1}}(t) = \begin{bmatrix} 0 & -t & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{P_{0,4}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{P_{0,1}}(t) = \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{P_{0,6}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2t & 0 \end{bmatrix},$$

$$\mathbf{P_{0,7}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{P_{1,2}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{bmatrix}, \mathbf{P_{1,4}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -t & 0 & 0 \end{bmatrix}, \mathbf{P_{1,6}}(t) = \begin{bmatrix} 2t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{P_{1,8}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ -t & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{P_{2,1}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{P_{2,3}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{P_{0,6}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\overline{\mathbf{M}}_{\mathbf{1,a}} = \mathbf{diag}\left[\mathbf{M_{1,a}},\, \mathbf{M_{1,a}},\, \mathbf{M_{1,a}}\right], \quad a = -{}^{1}\!/_{6}, -{}^{1}\!/_{2}, \pm{}^{1}\!/_{4}, \pm{}^{1}\!/_{3}, {}^{1}\!/_{5}, -1$$

$$\mathbf{M}_{1,-1/2} = \begin{bmatrix} 1 & -{}^{1}\!/_{2} & {}^{1}\!/_{4} & -{}^{1}\!/_{8} \\ 0 & 1 & -1 & {}^{3}\!/_{4} \\ 0 & 0 & 1 & -{}^{3}\!/_{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{M}_{1,-1/6} = \begin{bmatrix} 1 & -{}^{1}\!/_{6} & {}^{1}\!/_{36} & -{}^{1}\!/_{216} \\ 0 & 1 & -{}^{1}\!/_{3} & {}^{1}\!/_{12} \\ 0 & 0 & 1 & -{}^{1}\!/_{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{M}_{1,-1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{M}_{1,\pm 1/3} = \begin{bmatrix} 1 & \pm 1/3 & 1/9 & \pm 1/27 \\ 0 & 1 & \pm 2/3 & 1/3 \\ 0 & 0 & 1 & \pm 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{M}_{1,\pm 1/4} = \begin{bmatrix} 1 & \pm 1/4 & 1/16 & \pm 1/64 \\ 0 & 1 & \pm 1/2 & 3/16 \\ 0 & 0 & 1 & \pm 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{M}_{1,1/5} = \begin{bmatrix} 1 & 1/5 & 1/25 & 1/125 \\ 0 & 1 & 2/5 & 3/25 \\ 0 & 0 & 1 & 3/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \overline{\mathbf{T}}(0) \\ \overline{\mathbf{T}}\left({}^{1}\!/_{3}\right) \\ \overline{\mathbf{T}}\left({}^{2}\!/_{3}\right) \\ \overline{\mathbf{T}}(1) \end{bmatrix}, \overline{\mathbf{T}}(t_s) = \mathbf{diag}\left[\mathbf{T}(t_s),\, \mathbf{T}(t_s),\, \mathbf{T}(t_s)\right], \begin{array}{l} \mathbf{T}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{T}\left({}^{1}\!/_{3}\right) = \begin{bmatrix} 1 & {}^{1}\!/_{3} & {}^{1}\!/_{9} & {}^{1}\!/_{27} \end{bmatrix}, \\ \mathbf{T}\left({}^{2}\!/_{3}\right) = \begin{bmatrix} 1 & {}^{2}\!/_{3} & {}^{4}\!/_{9} & {}^{8}\!/_{27} \end{bmatrix}, \\ \mathbf{T}(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}. \end{array}$$

$$\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & 0 \\ 0 & \mathbf{B} & 0 \\ 0 & 0 & \mathbf{B} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ -30 & 10 & 0 & 0 \\ 60 & -30 & 6 & 0 \\ -35 & 21 & -7 & 1 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}(0) \\ \mathbf{g}(1/3) \\ \mathbf{g}(2/3) \\ \mathbf{g}(1) \end{bmatrix}, \mathbf{g}(0) = \begin{bmatrix} 501/1045 \\ 1975/957 \\ 2063/965 \end{bmatrix}, \mathbf{g}\left(\frac{1}{3}\right) = \begin{bmatrix} 1313/3140 \\ 803/245 \\ 5293/1282 \end{bmatrix}, \mathbf{g}\left(\frac{2}{3}\right) = \begin{bmatrix} 677/3141 \\ 247/48 \\ 1085/176 \end{bmatrix},$$

and

$$g(1) = \begin{bmatrix} 198/1745 \\ 2065/264 \\ 3581/427 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \mathbf{A_3} \end{bmatrix} \text{ and } \begin{array}{l} \mathbf{A_1} = \\ \mathbf{A_2} = \\ \mathbf{A_3} = \end{array} \begin{bmatrix} a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \\ a_{30} & a_{31} & a_{32} \end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is calculated as

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}
225 & -123 & 33 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 501/1045\\
0 & 0 & 0 & 0 & 345/2 & -183/2 & 45/2 & -3/2 & 1535/8 & -605/8 & 95/8 & -5/8 & ; & 1975/957\\
-57/64 & 183/64 & 51/64 & 3/64 & 0 & 0 & 0 & 0 & 50 & -18 & -2 & 2 & ; & 2063/965\\
1415/9 & -247/3 & 163/9 & -1/9 & -2339/648 & 187/216 & -43/648 & 1/648 & 1648/243 & -236/81 & 128/243 & -8/243 & ; & 1313/3140\\
-62/45 & 34/25 & -32/225 & -64/225 & 66311/648 & -10759/216 & 5479/648 & 323/648 & 11695/216 & -935/72 & 215/216 & -5/216 & ; & 803/245\\
809/192 & -77/64 & 95/64 & 109/192 & 16/81 & -8/27 & 32/81 & 16/81 & -30 & 82/3 & -16 & 4 & ; & 5293/1282\\
65 & -65/3 & -7 & 5 & -109/324 & -67/108 & -29/324 & -1/324 & 2372/243 & -268/81 & 100/243 & -4/243 & ; & 677/3141\\
146/45 & -266/75 & 806/225 & -338/225 & 131/4 & -91/12 & -71/12 & 29/12 & 545/216 & 335/72 & 145/216 & 5/216 & ; & 247/48\\
-1513/576 & 207/64 & -893/576 & 1043/576 & -4/3 & 4/3 & -4/3 & 4/3 & -820/9 & 64 & -260/9 & 56/9 & ; & 1085/176\\
-295/3 & 87 & -155/3 & 41/3 & 3/8 & -1/8 & -5/8 & -1/8 & 4 & 0 & 0 & 0 & ; & 198/1745\\
186/5 & -718/25 & 396/25 & -108/25 & -8189/216 & 2509/72 & -4237/216 & 847/216 & -15/8 & 5/8 & 25/8 & 5/8 & ; & 2065/264\\
-4111/192 & 1035/64 & -1595/192 & 725/192 & -664/27 & 176/9 & -320/27 & 128/27 & -470/3 & 106 & -136/3 & 28/3 & ; & 3581/427
\end{bmatrix}$$

From Eq.(19), the matrix form for initial conditions is computated as

$$[\mathbf{U},\alpha]=\begin{bmatrix}
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & 0\\
-30 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1\\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1\\
0 & 0 & 0 & 0 & -30 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & ; & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -30 & 10 & 0 & 0 & ; & 1
\end{bmatrix}$$

Hence, the new augmented matrix based on conditions from systems (20) can be obtained as follows

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix}
225 & -123 & 33 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 501/1045\\
0 & 0 & 0 & 0 & 345/2 & -183/2 & 45/2 & -3/2 & 1535/8 & -605/8 & 95/8 & -5/8 & ; & 1975/957\\
-57/64 & 183/64 & 51/64 & 3/64 & 0 & 0 & 0 & 0 & 50 & -18 & -2 & 2 & ; & 2063/965\\
1415/9 & -247/3 & 163/9 & -1/9 & -2339/648 & 187/216 & -43/648 & 1/648 & 1648/243 & -236/81 & 128/243 & -8/243 & ; & 1313/3140\\
-62/45 & 34/25 & -32/225 & -64/225 & 66311/648 & -10759/216 & 5479/648 & 323/648 & 11695/216 & -935/72 & 215/216 & -5/216 & ; & 803/245\\
809/192 & -77/64 & 95/64 & 109/192 & 16/81 & -8/27 & 32/81 & 16/81 & -30 & 82/3 & -16 & 4 & ; & 5293/1282\\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0\\
-30 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1\\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1\\
0 & 0 & 0 & 0 & -30 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & ; & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -30 & 10 & 0 & 0 & ; & 1
\end{bmatrix}$$

By solving this system, substituting the resulting unknown Chelyshkov coefficients matrix into Eq.(6) we obtain the approximation solutions for $N=3$ as

$$\begin{aligned}
y_{1,3}(t) &= -0.156386235379089t^3+0.005133416233467t^2+t,\\
y_{2,3}(t) &= -0.097456081339467t^3-0.464896366222704t^2+1,\\
y_{1,3}(t) &= 0.279477737475736t^3+0.417618458568886t^2+t+1.
\end{aligned}$$

By the help of similar process for $N=5,\ 8$ we get the approximate solutions, respectively, as

$$\begin{aligned}
y_{1,5}(t) &= 0.008078454628850t^5-0.000339763804079t^4-0.166614138413128t^3+0.000399275674599t^2+t,\\
y_{2,5}(t) &= -0.005373175038781t^5+0.047460754150710t^4-0.002908215549952t^3-0.499185094547048t^2+1,\\
y_{3,5}(t) &= 0.015287450418155t^5+0.031198830705710t^4+0.17387878980801t^3+0.497429866377082t^2+t+1,
\end{aligned}$$

and

$$y_{1,8}(t) = -0.00000321590510t^8 - 0.00018333553615t^7 + 0.000005602017367t^6$$
$$+0.00831597785561t^5 - 0.0000075041566t^4 - 0.1666612290552t^3 + 0.0000029953989t^2 + t,$$
$$y_{2,8}(t) = 0.000004923437t^8 + 0.0000461166447t^7 - 0.00143061106296t^6$$
$$+0.000023279619900t^5 + 0.0416575434086t^4 - 4.569431\ 10^{-7}t^3 - 0.49999934365088t^2 + 1,$$
$$y_{3,8}(t) = 0.000050736208t^8 + 0.00012301814685t^7 + 0.001494329017229t^6$$
$$+0.0082608026944t^5 + 0.416841051499t^4 + 0.1666722645286t^3 + 0.4999953383022t^2 + 1.$$

Tables 1-3 show the comparison of some numerical values of the absolute errors of Chelyshkov polynomial solutions for $N = 3, 5, 8$ and 11 , and also Figs 1a, 1b and 1c display the exact and approximate solutions of Eq. (22). From tables, we see that the errors decrease rapidly as N increases.

Table 1: Comparisons of the absolute error functions $E_{1,N}(t)$ of Eq. (22).

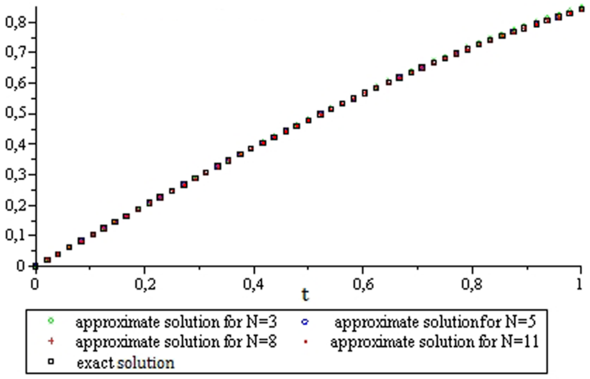
t_i	N=3	N=5	N=8	N=11
	$\lvert e_{1,3}(t_i)\rvert$	$\lvert e_{1,5}(t_i)\rvert$	$\lvert e_{1,8}(t_i)\rvert$	$\lvert e_{1,11}(t_i)\rvert$
0,1	4,868153e-3	6,500758e-5	2,858860e-8	6,336130e-10
0,3	1,922352e-3	2,566262e-5	1,134936e-8	1,181050e-10
0,5	1,134043e-2	2,282453e-5	1,038000e-12	2,300859e-9
0,7	1,237158e-2	3,069425e-5	8,407349e-8	1,135740e-8
0,9	3,262414e-2	8,356441e-4	7,440406e-6	1,591974e-7

Table 2: Comparisons of the absolute error functions $E_{2,N}(t)$ of Eq. (22).

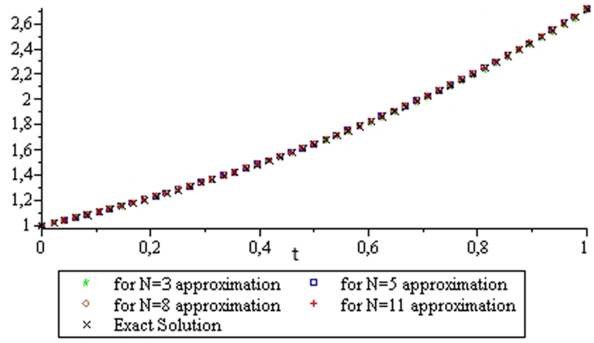
t_i	N=3	N=5	N=8	N=11
	$\lvert e_{2,3}(t_i)\rvert$	$\lvert e_{2,5}(t_i)\rvert$	$\lvert e_{2,8}(t_i)\rvert$	$\lvert e_{2,11}(t_i)\rvert$
0,1	3,918014e-2	8,732413e-5	4,219270e-9	1,11890e-10
0,3	1,521920e-2	4,580208e-5	1,388750e-9	3,88928e-10
0,5	1,131821e-1	6,625824e-5	4,100000e-12	1,77634e-9
0,7	3,044190e-1	4,012964e-4	5,188580e-9	4,54605e-9
0,9	5,121444e-1	3,136185e-3	2,056825e-7	1,99394e-9

Table 3: Comparisons of the absolute error functions $E_{3,N}(t)$ of Eq. (22).

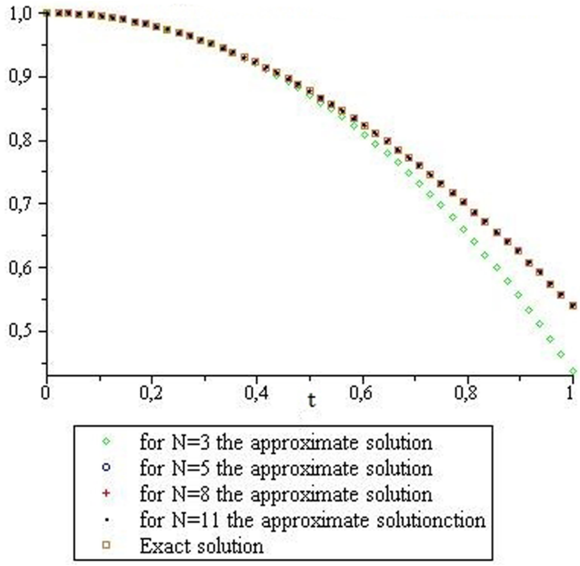
t_i	N=3	N=5	N=8	N=11
	$\lvert e_{3,3}(t_i)\rvert$	$\lvert e_{3,5}(t_i)\rvert$	$\lvert e_{3,8}(t_i)\rvert$	$\lvert e_{3,11}(t_i)\rvert$
0.1	2,794883e-2	1,857134e-4	1,611955e-8	4,598110e-9
0.3	1,272921e-2	1,083602e-4	6,474959e-9	2,462238e-8
0.5	1,167859e-1	1,767717e-4	2,485630e-10	1,226285e-7
0.7	4,076591e-1	1,219935e-3	5,036500e-8	4,906619e-7
0.9	9,376063e-1	1,091131e-2	4,402593e-6	1,618757e-6



(a) Comparison of the exact solution $\sin t$ and the approximate solutions $y_{1,N}(t)$



(b) Comparison of the exact solution e^t and the approximate solutions $y_{3,N}(t)$



(c) Comparison of the exact solution $\cos t$ and the approximate solutions $y_{2,N}(t)$

Fig. 1: Graphs of exact and numerical solutions of equation (22) for $N = 3,5,8$ and 11.

Example 2. [24]. Let us consider system of the linear differential difference equations with variable coefficients given by

$$\begin{cases} y_1{}^{(2)}(t) - y_1(t) + y_2(t) - y_1(t-0.2) = -e^{t-0.2} + e^{-t} \\ y_2{}^{(2)}(t) + y_1(t) - y_2(t) - y_2(t-0.2) = e^{-t+0.2} + e^{t}, \end{cases} \quad 0 \le t \le 1 \tag{23}$$

with the initial conditions $y_1(0) = 1, y_1'(0) = 1, y_2(0) = 1, y_2'(0) = -1$ and the exact solutions $y_1(t) = e^t, y_2(t) = e^{-t}$. Here,

$$Q^0_{1,1}(t) = -1, Q^0_{1,2}(t) = 1, Q^0_{2,1}(t) = 1, Q^0_{2,2}(t) = -1, Q^2_{1,1}(t) = 1, Q^2_{1,2}(t) = 0 = Q^2_{2,1}(t), Q^2_{2,2}(t) = 1,$$
$$P^{0,0}_{1,1}(t) = -1, P^{0,0}_{1,2}(t) = 0, P^{0,0}_{2,1}(t) = 0, P^{0,0}_{2,2}(t) = -1, g_1(t) = -e^{t-0.2} + e^{-t} \text{and} g_2(t) = e^{-t+0.2} + e^{t}.$$

From Eq.(16), the fundamental matrix equation of the problem is

$$\{\mathbf{Q_0}\ \mathbf{T}\overline{\mathbf{C}} + \mathbf{P_{00}}\mathbf{T}\overline{\mathbf{M}}_{1,-0.2}\overline{\mathbf{C}} + \mathbf{Q_2}\mathbf{T}\overline{\mathbf{B}}^2\overline{\mathbf{C}}\}\mathbf{A} = \mathbf{G}.$$

Using the procedure in Section 3, we get the approximate solutions by the Chelyshkov polynomials of the problem for $N = 3, 6, 10$

$$y_{1,3}(t) = -0.158267390000000t^3 + t,$$
$$y_{2,3}(t) = 0.031898368270000t^3 - 0.54t^2 + 1,$$

$$y_{1,6}(t) = -0.000969835545673t^6 + 0.007944669136359t^5 - 0.000472361651851t^4$$
$$-0.173874073649937t^3 + t,$$
$$y_{2,6}(t) = 0.004574729687886t^6 + 0.003147215190958t^5 + 0.040559771740590t^4$$
$$-0.007110146730438t^3 - 0.5t^2 + 1,$$

and

$$y_{1,10}(t) = -0.000107408790171t^{10} + 0.000326131441025t^9 - 0.000484626701101t^8$$
$$+0.00165892945991t^7 - 0.000163899026982t^6 + 0.007972064081813t^5$$
$$-0.000007259719873t^4 - 0.168699403354326t^3 + t,$$
$$y_{2,10}(t) = -0.000512184619072t^{10} + 0.001150293165116t^9 - 0.001026116532671t^8$$
$$+0.000310064322725t^7 - 0.0016848497042t^6 + 0.000180118015455t^5$$
$$+0.041654471237032t^4 - 0.002032382364153t^3 - 0.5t^2 + 1.$$

Table 4: Numerical results of the exact and the approximate solutions $y_{1,N}(t)$ for N=3,6,10.

t_i	Exact Value	Approximation solutions		
	e^{t_i}	$y_{1,3}(t_i)$	$y_{1,6}(t_i)$	$y_{1,10}(t_i)$
0.1	1.10517091807565	1.1051964699464	1,105170903152	1,10517091807564
0.3	1.34985880757600	1.3503336525308	1,349858707756	1,34985880757562
0.5	1.64872127070013	1.6497196542882	1,648721089989	1,64872127070873
0.7	2.01375270747048	2.0128622699197	2,013752462911	2,01375270751456
0.9	2.45960311115695	2.4492692941264	2,459601077725	2,45960311132513

Table 5: Numerical results of the exact and the approximate solutions $y_{2,N}(t)$ for N=3,6,10.

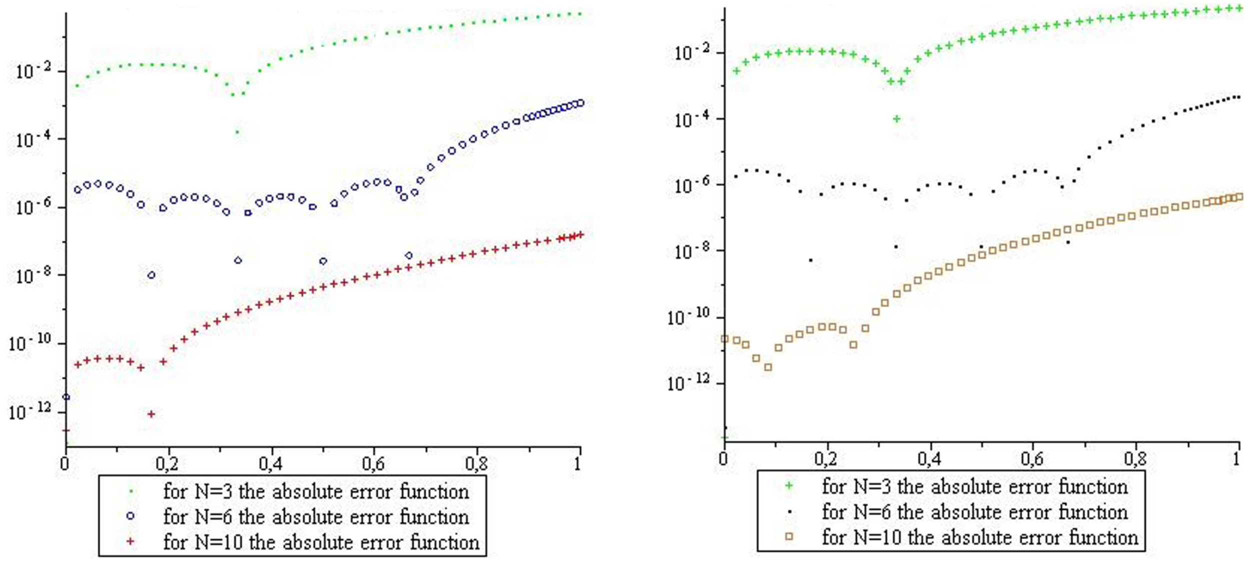
t_i	Exact Value	Approximation solutions		
	e^{-t_i}	$y_{2,3}(t_i)$	$y_{2,6}(t_i)$	$y_{2,10}(t_i)$
0.1	0.9048374180359	0.904856967280	9,048374264143	0,904837418036002
0.3	0.7408182206817	0.741162866226	7,408182774021	0,740818220680527
0.5	0.6065306597126	0.607258408138	6,065307651909	0,606530659695493
0.7	0.4965853037914	0.496344021556	4,965854570327	0,496585303678594
0.9	0.4065696597405	0.401620135023	4,065706243200	0,406569659073933

Table 6: The maximum errors $E_{1,N}(t)$ and $E_{2,N}(t)$ of Eq. (23).

N	3	6	10
$E_{1,N}(t)$	1,0334e-2	2,0334e-6	1,6818e-10
$E_{1,N}(t)$	4,9495e-3	9,6458e-7	6,6667e-10

In Tables 4 and 5, it is given a comparison of numerical results of the approximate solutions obtained by the presented method for $N = 3, 6$ and 10 with the exact solutions of Eq. (23). In addition, the absolute error functions are shown in

Fig. 2. As seen from Table 6, the resulting solutions from Chelyshkov polynomial method for $N = 10$ are almost the same as the results of the exact solutions and we see that the errors decrease rapidly as N increases.



(a) Comparison of the absolute errors functions $e_{1,N}(t)$ for $N = 3,6$ and 10

(b) Comparison of the absolute errors functions $e_{2,N}(t)$ for $N = 3,6$ and 10

Fig. 2: The Graph of the function in equation (23) for $N = 3,6$ and 10.

Example 3. [25]. Let us consider the system of initial value problems given by

$$\begin{aligned} &y_1'(t-1)+y_2'(t-1)=2t, \qquad y_1(0)=0\\ &y_1'(t-1)-y_3'(t-1)=2t-1,\ y_2(0)=0,\ 0\le t\le 1\\ &y_1'(t-1)+y_3(t-1)=t-1, \quad y_3(0)=0 \end{aligned} \tag{24}$$

Using the present method for $N = 3$ as in Example 1, we obtain solutions of the problem as $y_1(t) = t^2, y_2(t) = 2t$ and $y_3(t) = -t$ which are the exact solutions. Moreover, if higher values of N be chosen, we obtain the exact solution again.

5 Conclusions

In this paper, we have presented a new collocation method and used it for the systems of the mentioned linear functional differential equations with variable coefficients. The comparison of the results shows that the present method is a powerful mathematical tool for finding the numerical solutions of these type systems. One of the considerable advantages of the method is that the approximate solutions are found very easily by using avaible software such as maple or matlab since the method is based on matrix operations. Moreover, the method proposed in this work can be extended to solve the systems of nonlinear equations which play an important role in physics and engineering.

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