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Solution of the two-dimensional heat equation for a rectangular plate

Emel Kurul[1] *and Nurcan Baykus Savasaneril*[2]

[1]Department of Mathematics, Faculty of Sciences, Selcuk University, Konya, Turkey
[2] Izmir Vocational School, Dokuz Eylul University, Izmir, Turkey

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Abstract: Laplace equation is a fundamental equation of applied mathematics. Important phenomena in engineering and physics, such as steady-state temperature distribution, electrostatic potential and fluid flow, are modeled by means of this equation. The Laplace equation which satisfies boundary values is known as the Dirichlet problem. The solutions to the Dirichlet problem form one of the most celebrated topics in the area of applied mathematics. In this study, a novel method is presented for the solution of two-dimensional heat equation for a rectangular plate. In this alternative method, the solution function of the problem is based on the Green function, and therefore on elliptic functions.

Keywords: Heat equation, Dirichlet problem, elliptic functions, elliptic integral, green function.

1 Introduction

Laplace's equation is one of the most significant equations in physics. It is the solution to problems in a wide variety of fields including thermodynamics and electrodynamics. Today, the theory of complex variables is used to solve problems of heat flow, fluid mechanics, aerodynamics, electromagnetic theory and practically every other field of science and engineering. A broad class of steady-state physical problems can be reduced to finding the harmonic functions that satisfy certain boundary conditions. The Dirichlet problem for the Laplace equation is one of the above mentioned problems.

The Dirichlet problem is to find a function $U(z)$that is harmonic in a bounded domain $D\subset R^2$, is continuous up to the boundary $\partial D$of D, assumes the specified values $U_0(z)$ on the boundary ∂D, where $U_0(z)$ is a continuous function on ∂D, and can be formulated as

$$\nabla^2 U=0,\quad z\in D,\quad U\big|_{z\in\partial D}=U_0(z) \tag{1}$$

Here, for a point (x,y) in the plane R^2, one takes the complex notation $z=x+iy$, $U(z)=U(x,y)$ and $U_0(z)=U_0(x,y)$ are real functions and $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator. Similarly the Dirichlet problem for the Poisson equation can be formulated as

$$\nabla^2 U=h(z),\quad z\in D,\quad U\big|_{z\in\partial D}=U_0(z) \tag{2}$$

Brovar et al. [1] investigated the relation between the Dirichlet problem and the Cauchy problem. Sezer [2] developed a new method for the solution of Dirichlet problem. Lanzara [3] studied Dirichlet problem for second degree elliptic linear

* Corresponding author e-mail: emelkurul@gmail.com: nurcan.savasaneril@deu.edu.tr

equations with limited and measurable coefficients. The dependence upon variations of problem data of the solution of two-dimensional Dirichlet boundary value problem for simply connected regions was investigated [4]. Han and Hasebe [5] derived Green's function for a thermomechanical mixed boundary value problem of an infinite plane with an elliptic hole under a pair of heat source and sink. In another study, Han and Hasebe [6] also reviewed Green's functions for a point heat source in various thermoelastic boundary value problems for an infinite plane with an inhomogeneity. Green function of the Dirichlet problem for the Laplace differential equation in a rectangular domain was expressed in terms of elliptic functions and the solution of the problem was based on the Green function and therefore on elliptic functions by Kurt et al. [7]. Hsiao and Saranen [8] showed an equivalence between the weak solution and the various boundary integral solutions, and described a coupling procedure for an exterior initial boundary value problem for the nonhomogeneous heat equation. The problem of the one-dimensional heat equation with nonlinear boundary conditions was studied by Tao [9]. Hansen [10] studied a boundary integral method for the solution of the heat equation in an unbounded domain D in R^2. The application of spectral methods for solving the one-dimensional heat equation was presented by Saldana et al. [11]. Al-Najem et al. [12] estimated the surface temperature in two-dimensional steady-state in a rectangular region by two different methods, the singular value decomposition with boundary element method and the least-squares approach with integral transform method. The Green function of the Dirichlet problem for the Laplace differential equation in a triangle region was expressed in terms of elliptic functions and the solution of problem was based on the Green function, and therefore on elliptic functions by Kurt and Sezer [13]. Green function of the two-dimensional heat equation in a square region was expressed in terms of elliptic functions and the solution of the problem was based on the Green function and therefore on elliptic functions by Kurt [14]. Least Square Method (LSM), Collocation Method (CM) and a new approach which is called Akbari-Ganji's Method (AGM) are applied to solve the nonlinear heat transfer equation of fin with power-law temperature-dependent both thermal conductivity and heat transfer coefficient by Ledari et al. [15].

As it is known, the solution of the Dirichlet problem by the method of separation of variables may be obtained only for a restricted class of domains D with a sufficiently simple boundary ∂D. The conformal mappings yield a sufficiently universal algorithm for the solution of the Dirichlet problem for two-dimensional domains. These permit constructing a Green function of the Dirichlet problem for the Laplace (and Poisson) equation in a Dconformally mapped onto the unit circle or upper half-plane, and cannot be obtained in terms of elliptic functions.

Our purpose in this paper is, first, to determine the analytic function which maps the rectangular domain D onto the upper half-plane or the unit circle in terms of elliptic functions using the Schwarz-Christoffel transformation and conformal mapping, and then, to find the solution of the Dirichlet problem for the rectangular domain in terms of elliptic functions, by means of the relation between the obtained analytic function and the Green function.

2 Elliptic integrals and functions

The integral

$$\int_0^t \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = \int_0^u du_1 = u = sn^{-1}(t,k) = F(\phi,k) \;\; t = \sin\phi \tag{3}$$

is called the normal elliptic integral of the first kind, where k, $(0<k<1)$is any number. When $t=1$, Eq. (3) is said to be complete and becomes

$$\int_0^t \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = \int_0^K du_1 = F\left(\frac{\pi}{2},k\right) \equiv K(k) \equiv K \tag{4}$$

or

$$\int_0^t \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = \int_0^K du_1 = F\left(\frac{\pi}{2},k'\right) \equiv K'(k) \equiv K(k') \equiv K' \tag{5}$$

Here, the number k is the modulus and k', $(0 < k' < 1)$ is the complementary modulus, such that $k'^2 = 1-k^2$. If $k = 0$ in Eq. (3), one finds that $u = \sin^{-1} t$ or $t = \sin u$. When$k \neq 0$, the integral (3) is denoted by $u = sn^{-1}(t,k)$ or briefly $u = sn^{-1}t$ or $t = sn\, u$. The function $sn\, u$ is called Jacobian elliptic function. Two other Jacobian elliptic functions can be defined by $cn(u,k) = (1-k^2)^{1}/2.$, $dn(u,k) = (1-k^2sn^2u)^{1}/2$.

3 The conform mapping of a rectangular domain

Let apply the transformation $z_1 = z+a+ib$ to carry the rectangle onto the first quadrant. The function$\zeta_1 = sn\lambda z$, with $\lambda = K/2a = K'/2b$, maps the rectangle onto the first quadrant of the ζ_1-plane. The function $\zeta = \zeta_1^2$ maps this quadrant onto the upper half of the $\zeta-$plane. The following table shows where the corresponding points lie in different planes (including the final w-plane):

plane	A	B	C	D
z	a	ib	$-a$	$-ib$
z_1	$a+2ib$	$a+2ib$	ib	a
λz_1	$K+iK'/2$	$K/2+iK'$	$iK'/2$	$K/2$
ζ_1	$1/\sqrt{k}$	$\sqrt{1+k'}/k$	$i/\sqrt{k}$	$1/\sqrt{1+k'}$
ζ	$1/k$	$(1+k'^2$	$-1/k$	$1/(1+k')$
w	1	i	-1	$-i$

The last two rows define the bilinear transformation between the $\zeta-$ and the $w-$planes:

$$\frac{w-1}{w+1} = -i\frac{k+1+k'}{k-1-k'}\,\frac{k\zeta-1}{k\zeta+1} \tag{6}$$

That is,

$$\frac{w-1}{w+1} = i\sqrt{\frac{1+k}{1-k}}\,\frac{k\zeta-1}{k\zeta+1} \tag{7}$$

On the other hand,

$$\begin{aligned}\zeta &= sn^2\left(\lambda z+\tfrac{K}{2}+i\tfrac{K'}{2}\right) = sn^2\left[\tfrac{1}{2}(2\lambda z+K+iK')\right]\\ &= \tfrac{1-cn(2\lambda z+K+iK')}{1+dn(2\lambda z+K+iK')} = \tfrac{1+i(k'k)nc2\lambda z}{1+ik'sc2\lambda z}\end{aligned} \tag{8}$$

Hence,

$$\frac{w-1}{w+1} = i\frac{-\sqrt{1-k}cn2\lambda z+i\sqrt{1+k}(1-sn2\lambda z)}{\sqrt{1+k}cn2\lambda z+i\sqrt{1-k}(1+sn2\lambda z)}$$

$$w = \frac{\sqrt{1+k}(cn2\lambda z+sn2\lambda z-1)+i\sqrt{1-k}(sn2\lambda z-cn2\lambda z+1)}{\sqrt{1+k}(cn2\lambda z-sn2\lambda z+1)+i\sqrt{1-k}(sn2\lambda z+1)} \tag{9}$$

Multiply the numerator and denominator by the denominator with $-i$ instead of i (not the conjugate!). Using $1-k^2=k'^2$ and $sn^2+cn^2=1$ and simplifying, we obtain:

$$\begin{aligned} w &= \frac{sn2\lambda z}{1+cn2\lambda z} = \frac{2sn\lambda z cn\lambda z dn\lambda z}{cn^2\lambda z+sn^2\lambda z dn^2\lambda z} \\ &= \frac{cn^2\lambda z+sn^2\lambda z dn^2\lambda z}{cn^2\lambda z+sn^2\lambda z dn^2\lambda z+cn^2\lambda z-sn^2\lambda z dn^2\lambda z} \\ &= \frac{sn\lambda z dn\lambda z}{cn\lambda z} = sc\lambda z dn\lambda z \end{aligned} \tag{10}$$

The conformal mapping of the rectangle given in the z-plane onto the unit circle $|w|<1$ in the $w-$plane can be written as

$$w=F(z)=sc\lambda z dn\lambda z \quad scz=\frac{snz}{cnz} \tag{11}$$

4 Determination of green function

Green function $G(z,\zeta)$ of the Dirichlet problem for the Laplace equation in the domain D is defined by

$$G(z,\zeta)=\frac{1}{2\pi}\ln|z-\zeta|+g(z,\zeta) \quad z\in D\,,\ \zeta\in D \tag{12}$$

where g is a harmonic function in D for each $\zeta\in D$ and $g(z,\zeta)=-1/(2\pi)\ln|z-\zeta|$ then $G(z,\zeta)=0$, for each $z\in\partial D$, $z=x+iy$ and $\zeta=\xi+i\eta$.

When the domain $D$is simply connected, the determination of the mentioned Green function can be reduced to the problem of determining an analytic function which specifies a mapping of D onto the upper half-plane $ImW>0$ or the unit circle $|w|<1$. This is because $W=F(z)$ is an analytic function which maps the domain D in the z-plane onto the upper half-plane of the $W-$plane, with $F'(z)\neq 0$ in D then the mapping is one-to-one.

$$G=\frac{1}{2}\ln\left|\frac{F(z)-F(\zeta)}{F(z)-\overline{F(\zeta)}}\right|\,,\quad z=x+iy\,,\ \zeta=\xi+i\eta \tag{13}$$

and, if the analytic function $W=F(z)$ maps D onto the unit circle $|w|<1$, then the Green function of the Dirichlet problem for the Laplace operator in D becomes

$$G(z,\zeta)=\frac{1}{2\pi}\ln|w(z-\zeta)|\ ,\ W(z,\zeta)=\frac{F(z)-F(\zeta)}{1-F(z)\overline{F(\zeta)}} \tag{14}$$

Consequently, if one takes D as the rectangle$A_1\left(\frac{K}{2\lambda},-\frac{K'}{2\lambda}\right)$,$A_2\left(\frac{K}{2\lambda},\frac{K'}{2\lambda}\right)$, $A_3\left(-\frac{K}{2\lambda},\frac{K'}{2\lambda}\right)$, $A_4\left(-\frac{K}{2\lambda},-\frac{K'}{2\lambda}\right)$ then from Eqs. (12) and (14), the Green function for the rectangle is found in the form

$$G=\frac{1}{2\pi}\ln\left|\frac{sc\lambda z dn\lambda z-sc\lambda\zeta dn\zeta z}{1-sc\lambda z dn\lambda z\,\overline{sc\lambda\zeta dn\zeta}}\right|=\frac{1}{2\pi}Re\ln\left[\frac{sc\lambda z dn\lambda z-sc\lambda\zeta dn\zeta z}{1-sc\lambda z dn\lambda z\,\overline{sc\lambda\zeta dn\zeta}}\right] \tag{15}$$

5 The solution of the Dirichlet problem

The solution of the Dirichlet problem for the Poisson equation (2) in D can be obtained as

$$U(z)=\int\int_D G(z,\zeta)h(\zeta)d\xi\,d\eta+\int_{\partial D}\frac{G(z,\zeta)}{\partial n}U_0(\zeta)\,|d\zeta| \tag{16}$$

where G is the Green function for the domain D and $\partial/\partial n$ denotes differentiation along an outward normal to the boundary ∂D of D with respect to ζ.

Taking the domain D as the rectangle $A_1\left(\frac{K}{2\lambda},-\frac{K'}{2\lambda}\right)$,$A_2\left(\frac{K}{2\lambda},\frac{K'}{2\lambda}\right)$, $A_3\left(-\frac{K}{2\lambda},\frac{K'}{2\lambda}\right)$, $A_4\left(-\frac{K}{2\lambda},-\frac{K'}{2\lambda}\right)$ and the boundary ∂D of D as the circumference $\partial D=\overline{A_4A_1}\cup\overline{A_1A_2}\cup\overline{A_2A_3}\cup\overline{A_3A_4}$, one may write the conditions

1. $\eta=-\frac{K'}{2\lambda}, \quad d\eta=0, \quad -\frac{K}{2\lambda}\leq\xi\leq\frac{K}{2\lambda}$ on $\overline{A_4A_1}$
2. $\xi=\frac{K}{2\lambda}, \quad d\xi=0, \quad -\frac{K'}{2\lambda}\leq\eta\leq\frac{K'}{2\lambda}$ on $\overline{A_1A_2}$
3. $\eta=\frac{K'}{2\lambda}, \quad d\eta=0, \quad -\frac{K}{2\lambda}\leq\xi\leq\frac{K}{2\lambda}$ on $\overline{A_2A_3}$
4. $\xi=-\frac{K}{2\lambda}, \quad d\xi=0, \quad -\frac{K'}{2\lambda}\leq\eta\leq\frac{K'}{2\lambda}$ on $\overline{A_3A_4}$

Thus, from Eq. (16), the solution of Eq. (2) in the above rectangle becomes

$$U(z)=\int_{-\frac{K'}{2\lambda}}^{\frac{K'}{2\lambda}}\int_{-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}G(z,\zeta)h(\zeta)d\xi\,d\eta-\int_{-\frac{K}{2\lambda}}^{-\frac{K}{2\lambda}}\left[G_\xi^2(z,\zeta)+G_\eta^2(z,\zeta)\right]^{1/2}U_0(\zeta)\Big|_{\eta=-\frac{K'}{2\lambda}}^{-\frac{K'}{2\lambda}}d\xi$$

$$+\int_{-\frac{K'}{2\lambda}}^{\frac{K'}{2\lambda}}\left[G_\xi^2(z,\zeta)+G_\eta^2(z,\zeta)\right]^{1/2}U_0(\zeta)\Big|_{\xi=-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}d\eta \tag{17}$$

In the case of $h(z)=0$, the solution of the Dirichlet problem for the Laplace differential equation (1) in the above rectangle is obtained in terms of elliptic functions as :

$$U(z)=\int_{-\frac{K'}{2\lambda}}^{\frac{K'}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0(\zeta)\Big|_{\xi=-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}d\eta-\int_{-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0(\zeta)\Big|_{\eta=-\frac{K'}{2\lambda}}^{\frac{K'}{2\lambda}}d\xi \tag{18}$$

where the Green function G is defined by

$$G=\frac{1}{2\pi}Re\ln\left[\frac{sc\lambda z dn\lambda z-sc\lambda\zeta\,dn\lambda\zeta}{1-sc\lambda z dn\lambda z\,\overline{sc\lambda\zeta\,dn\lambda\zeta}}\right],\quad z=x+iy,\ \zeta=\xi+i\eta$$

according to Eq. (14). The boundary values $K(k)$ and $K'(k)$ are the complete elliptic integrals and are tabulated for the range $k(0<k<1)$.

6 Illustrative application

In this section, an illustrative example is given. The boundary of a rectangular sheet of metal is kept at constant temperature 50^0C on the upper edge, 20^0C on the bottom edge, and 0^0C on the other two edges. After a sufficient period of time, the temperature inside the plate reaches an equilibrium distribution. This steady-state temperature distribution $U(x,y)$ is determined in this application. Since no heat sources are present in the plate, the steady-state temperature U must satisfy

$$U_{xx}(x,y)+U_{yy}(x,y)=0,\qquad\left(-\frac{K}{2\lambda}\leq x\leq\frac{K}{2\lambda},\quad-\frac{K'}{2\lambda}\leq y\leq\frac{K'}{2\lambda}\right) \tag{19}$$

The boundary conditions are

$$\begin{array}{lll} U\left(x,-\frac{K'}{2\lambda}\right)=20^0C, & U\left(x,\frac{K'}{2\lambda}\right)=50^0C, & -\frac{K}{2\lambda}\leq x\leq\frac{K}{2\lambda} \\ U\left(-\frac{K}{2\lambda},y\right)=U\left(\frac{K}{2\lambda},y\right)=0^0C, & & -\frac{K'}{2\lambda}\leq y\leq\frac{K'}{2\lambda} \end{array} \tag{20}$$

The solution of $U(x,y)$ with two real variables satisfying the above conditions by the method of separation variables is

$$U_s(x,y)=\frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\frac{\sinh(2k-1)\pi x/K}{\sinh(2(2k-1))\pi K'/K}\sin\frac{(2k-1)\pi y}{K}\right) \tag{21}$$

where $k=(n+1)/2,\ n=1,2,\ldots$ On the other hand, the solution of the

$$U_E(z)=\int_{-\frac{K'}{2\lambda}}^{\frac{K'}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0(\zeta)|_{\xi=-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}d\eta-\int_{-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0(\zeta)|_{\eta=-\frac{K'}{2\lambda}}^{\frac{K'}{2\lambda}}d\xi \tag{22}$$

$$\begin{aligned}=&\int_{-\frac{K'}{2\lambda}}^{\frac{K'}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0\left(\frac{K}{2\lambda},\eta\right)d\eta-\int_{-\frac{K'}{2\lambda}}^{\frac{K'}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0\left(-\frac{K}{2\lambda},\eta\right)d\eta\\ &-\int_{-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0\left(\xi,\frac{K'}{2\lambda}\right)U_0(\zeta)d\xi+\int_{-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}U_0\left(\xi,-\frac{K'}{2\lambda}\right)d\xi\end{aligned}$$

where

$$U\left(\xi,-\frac{K'}{2\lambda}\right)=20^0C,\quad U\left(\xi,\frac{K'}{2\lambda}\right)=50^0C,\quad U\left(-\frac{K}{2\lambda},\eta\right)=U\left(\frac{K}{2\lambda},\eta\right)=0^0C \tag{23}$$

Substituting the conditions (23) into Eq. (22) yields

$$U_E(z)=-50\int_{-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}d\xi+20\int_{-\frac{K}{2\lambda}}^{\frac{K}{2\lambda}}\left[G_\xi^2+G_\eta^2\right]^{1/2}d\xi$$

where

$$G=\frac{1}{2\pi}\ln\left|\frac{sc\lambda z\,dn\lambda z-sc\lambda\zeta\,dn\zeta z}{1-sc\lambda z\,dn\lambda z\,\overline{sc\lambda\zeta\,dn\zeta}}\right|$$

7 Discussion

The most significant advantage of present method is that the result is obtained in terms of elliptic functions; because expressing the result in terms of elliptic functions facilitates many physics and engineering problems. An alternative method is presented for the solution of two-dimensional heat equation for a rectangular plate. Following the way in the present paper, the heat equation in similar plates, namely the ellipse, square and polygon plates, can be solved in terms of elliptic functions; thus, a major contribution can be made to the solution of similar problems in physics and engineering.

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