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n-tupled fixed point theorems for weak-contraction in partially ordered complete G-metric spaces

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Received: 9 May 2015, Revised: 23 June 2015, Accepted: 28 June 2015
Published online: 14 November 2015

Abstract: In this paper, we prove some n-tupled coincidence point theorems for a pair of mappings in partially ordered complete G-metric spaces satisfying weakly contractive type conditions. Our results generalize the results of several authors. Also, we discuss an example to illustrate our results.

Keywords: Weakly-contractive mapping, mixed g-monotone property, n-tupled coincidence point, n-tupled fixed point, partially ordered complete G-metric spaces.

1 Introduction

The weak contraction condition in Hilbert Space was introduced by Alber and Gurerre - Delabriere ([23]). Later Rhoades ([4]) has shown that the result of Alber and Gurerre - Delabriere ([23]) in Hilbert Spaces is also true in a complete metric space. Rhoades ([4]) established the fixed point theorem in a complete metric space by using the following contractive condition:

A weakly contractive mapping $T : X \to X$ which satisfies the condition

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)), \tag{1}$$

where $x,y \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous ad nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Remark 1. In this above result if $\varphi(t) = (1-k)t$ where $k \in (0,1)$ then we obtain the condition (1.1) of Banach contraction condition.

In 1963, S. Gahler ([16], [18])introduced the notion of 2-metric space which is generalization of metric spaces. Dhage in his Ph.D. thesis [1992] introduce a new class of generalized metrics called D-metrics. But topological structure of D-metric spaces was incorrect. In 2006, Mustafa and Sims ([21])introduced a new notion of generalized metric space called G-metric spaces. Mustafa studied many fixed point results for a self mapping in G-metric spaces, one can see in ([10], [8], [6]). In 2006, Bhaskar and Lakshmikantham ([19]) established coupled fixed point results for mixed monotone operators in partially ordered metric spaces. Afterwards, Lakshmikantham and Ciric ([20]) had established coupled

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coincidence and coupled fixed point theorems for two mappings F and g where F has the mixed g-monotone property. Recently, Samet and vetro ([1]) extend the concept of coupled fixed point to higher dimensions by introducing the notion of fixed point of n-order (or n-tupled fixed point, where n is natural number greater than or equal to 2) and proved some n-tupled fixed point results in complete metric spaces. Imdad and Soliman ([11]) inspired by this and he introduced the concepts of n-tupled coincidence point and proved even tupled coincidence point theorems for nonlinear ϕ-contraction satisfying mixed g-monotone property. Mishra et al. ([28]-[30]) have discussed interesting results on fixed point theorems in partial metric spaces and other spaces with different type of contraction conditions. Some importance and applications of these type of fixed point theorems are discussed in ([24]-[27]).

2 Preliminaries

Throughout, this paper $(X,\preceq)$ denotes a partially ordered set with the partial order $\preceq$. Now we recall some definitions and results:

Definition 2.1. ([22]) Let X be a non-empty set and R^+ the set of non-negative real number. If the real valued function $G : X \times X \times X \to R^+$ satisfies the following properties:

1.$G(x,y,z) = 0$, if $x = y = z$;
2.$0 < G(x,x,y)$, for all x, y $\in X$ with $x \neq y$;
3.$G(x,x,y) \leq G(x,y,z)$, foll x, y, z $\in X$ with $z \neq y$;
4.$G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, symmetric in all three variables;
5.$G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all x, y, z , a $\in X$ (rectangular inequality).

Then the function G is called a G-metric on X and the pair (X,G) is called a G-metric space.

Definition 2.2. ([22]) Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $lim_{n,m\to\infty} G(x,x_n,x_m) = 0$ and one say that the sequence $\{x_n\}$ is G-convergent to x.

Thus, if $\{x_n\} \to x$ in a G-metric space (X,G), then for any $\varepsilon > 0$, there exists a positive integer N such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$.

It was shown in ([22]) that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdroff, a sequence can converge at most to one point.

Definition 2.3. ([22]) Let (X,G) be a G-metric space. A sequence $\{x_n\}$ in X called $G-Cauchy$ if for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq N$, that is , if $G(x_n,x_m,x_l) \to 0$, as $n,m,l \to \infty$.

Lemma 2.4. ([22]) Let (X,G) be a G-metric space, then the following are equivalent;

1.$\{x_n\}$ is G-convergent to x;
2.$G(x_n,x_n,x) \to 0$, as n $\to \infty$;
3.$G(x_n,x,x) \to 0$, as n $\to \infty$;
4.$G(x_n,x_m,x) \to 0$, as n, m $\to \infty$.

Lemma 2.5. ([21]) Let (X,G) be a G-metric space, then the following are equivalent;

1.The sequence $\{x_n\}$ is G-Cauchy;
2.For every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$, for all n, m $\geq N$.;

Combining Lemma 2.4 and 2.5 we have the following result.

Lemma 2.6. ([6]) Let (X, G) be a G-metric space then $\{x_n\}$ is a G-Cauchy sequence if and only if for every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m > n \geq N$.

Definition 2.7. ([22]) A G-metric space (X, G) is called symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.8. ([22]) A G-metric space (X, G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence in (X, G) is convergent in X.

Definition 2.9. ([22]) Let X be a nonempty set. Then $(X, G, \preceq)$ is called an ordered G-metric space if;

1. (X, G) is metric space.
2. $(X, \preceq)$ is a partially ordered set.

Definition 2.10. ([9]) Let $(X, \preceq)$ is a partially ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.11. ([9]) Let (X, G) be a G-metric space. A mapping $F : X\times \to X$ is said to continuous if for any two sequence $\{x_n\}$ and $\{y_n\}$ G-convering to x and y respectively, $\{F(x_n, y_n)\}$ is G- convergent to $F(x, y)$.

In this paper, we use the new definitions of n-tupled coincidence point given by Imdad ([11]) and n-tupled fixed point given by Samet and Vetro([1]). Throughout the paper, we consider n to be even integer. Now we recall some basic concepts and definition:

Definition 2.12. ([11]) An element $(x^1, x^2, x^3 ....x^n)$ is called n n-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{aligned}
&F(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}) = x^{(1)} \\
&F(x^{(2)}, x^{(3)}, \ldots, x^{(n)}, x^{(1)}) = x^{(2)} \\
&F(x^{(3)}, \ldots, x^{(n)}, x^{(1)}, x^{(2)}) = x^{(3)} \\
&\vdots \\
&F(x^{(n)}, x^{(1)}, \ldots, x^{(n-2)}, x^{(n-1)}) = x^{(n)}.
\end{aligned}$$

Definition 2.13. ([11]) Let $(X, \preceq)$ is a partially ordered set and $F : X^n \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is non-decreasing in its odd position arguments and non-increasing in its even position arguments, that is, if

1. for all $x_1^{(1)}, x_2^{(1)} \in X$, $x_1^{(1)} \preceq x_2^{(1)} \Rightarrow F(x_1^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}) \preceq F(x_2^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)})$
2. for all $x_1^{(2)}, x_2^{(2)} \in X$, $x_1^{(2)} \preceq x_2^{(2)} \Rightarrow F(x^{(1)}, x_2^{(2)}, x^{(3)}, \ldots, x^{(n)}) \preceq F(x^{(1)}, x_1^{(2)}, x^{(3)}, \ldots, x^{(n)})$
3. for all $x_1^{(3)}, x_2^{(3)} \in X$, $x_1^{(3)} \preceq x_2^{(3)} \Rightarrow F(x^{(1)}, x^{(2)}, x_3^{(3)}, \ldots, x^{(n)}) \preceq F(x^{(1)}, x^{(2)}, x_2^{(3)}, \ldots, x^{(n)})$

$\vdots$

4. for all $x_1^{(3)}, x_2^{(n)} \in X$, $x_1^{(n)} \preceq x_2^{(n)} \Rightarrow F(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x_2^{(n)}) \preceq F(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x_1^{(n)})$.

Definition 2.14. ([11]) Let $(X,\preceq)$ is a partially ordered set and $F : X^n \to X$ and $g : X \to X$ be two mapping. Then the mapping F is said to have the g-mixed monotone property if F is g-nondecreasing in its odd position arguments and g-nonincreasing in its even position arguments, that is, if,

1. for all $x_1^{(1)},x_2^{(1)} \in X$, $gx_1^{(1)} \preceq gx_2^{(1)} \Rightarrow F(x_1^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)}) \preceq F(x_2^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)})$
2. for all $x_1^{(2)},x_2^{(2)} \in X$, $gx_1^{(2)} \preceq gx_2^{(2)} \Rightarrow F(x^{(1)},x_2^{(2)},x^{(3)},\ldots,x^{(n)}) \preceq F(x^{(1)},x_1^{(2)},x^{(3)},\ldots,x^{(n)})$
3. for all $x_1^{(3)},x_2^{(3)} \in X$, $gx_1^{(3)} \preceq gx_2^{(3)} \Rightarrow F(x^{(1)},x^{(2)},x_3^{(3)},\ldots,x^{(n)}) \preceq F(x^{(1)},x^{(2)},x_2^{(3)},\ldots,x^{(n)})$

   $\vdots$

4. for all $x_1^{(3)},x_2^{(n)} \in X$, $gx_1^{(n)} \preceq gx_2^{(n)} \Rightarrow F(x^{(1)},x^{(2)},x^{(3)},\ldots,x_2^{(n)}) \preceq F(x^{(1)},x^{(2)},x^{(3)},\ldots,x_1^{(n)})$.

Definition 2.15. ([11]) An element $(x^1,x^2,x^3.........x^n)$ is called an n-tupled coincidence point of the mapping $F : X^n \to X$ and $g : X \to X$ if

$$\begin{aligned}
&F(x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)}) = gx^{(1)}\\
&F(x^{(2)},x^{(3)},\ldots,x^{(n)},x^{(1)}) = gx^{(2)}\\
&F(x^{(3)},\ldots,x^{(n)},x^{(1)},x^{(2)}) = gx^{(3)}\\
&\vdots\\
&F(x^{(n)},x^{(1)},\ldots,x^{(n-2)},x^{(n-1)}) = gx^{(n)}.
\end{aligned}$$

Definition 2.16. ([11]) An element $(x^1,x^2,x^3.........x^n)$ is called an n-tupled fixed point of the mapping $F : X^n \to X$ and $g : X \to X$ if

$$\begin{aligned}
&F(x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)}) = gx^{(1)} = x^{(1)}\\
&F(x^{(2)},x^{(3)},\ldots,x^{(n)},x^{(1)}) = gx^{(2)} = x^{(2)}\\
&F(x^{(3)},\ldots,x^{(n)},x^{(1)},x^{(2)}) = gx^{(3)} = x^{(3)}\\
&\vdots\\
&F(x^{(n)},x^{(1)},\ldots,x^{(n-2)},x^{(n-1)}) = gx^{(n)} = x^{(n)}.
\end{aligned}$$

Definition 2.17. Let (R,d) be a partially ordered metric space under natural setting and let $F : R^n \to R$ be mapping defined by $F(x^{(1)},x^{(2)},...,x^{(n)}) = \frac{x^{(1)}+x^{(2)}+...+x^{(n)}}{n}$, for any $x^{(1)},x^{(2)},...,x^{(n)} \in R$ while $g : R \to R$ is defined as $g(x) = \frac{x}{3}$. Then $(0,0,0,...,0)$ is an n-tupled coincidence point of F and g.

Definition 2.18. Let $F : X^n \to X$ and $g : X \to X$ be two mapping. Then F is said be g-compatible if

$$\begin{aligned}
&lim_{m\to\infty} G(g(F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}),F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)})) = 0\\
&lim_{m\to\infty} G(g(F(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)}),F(gx_m^{(2)},\ldots,gx_m^{(n)},gx_m^{(1)}),F(gx_m^{(2)},\ldots,gx_m^{(n)},gx_m^{(1)})) = 0\\
&\vdots\\
&lim_{m\to\infty} G(g(F(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}),F(gx_m^{(n)},gx_m^{(1)},\ldots,gx_m^{(n-1)}),F(gx_m^{(n)},gx_m^{(1)},\ldots,gx_m^{(n-1)})) = 0.
\end{aligned}$$

where $x_m^{(1)}$, $x_m^{(2)}$,..., $x_m^{(n)}$ are the sequences in X such that

$$\begin{aligned}
&lim_{m\to\infty}F(x_m^{(1)},x_m^{(2)},x_m^{(3)},\ldots,x_m^{(n)}) = lim_{m\to\infty}gx_m^{(1)} = x^{(1)}\\
&lim_{m\to\infty}F(x_m^{(2)},x_m^{(3)},\ldots,x_m^{(n)},x_m^{(1)}) = lim_{m\to\infty}gx_m^{(2)} = x^{(2)}\\
&lim_{m\to\infty}F(x_m^{(3)},\ldots,x_m^{(n)},x_m^{(1)},x_m^{(2)}) = lim_{m\to\infty}gx_m^{(3)} = x^{(3)}\\
&\vdots\\
&lim_{m\to\infty}F(x_m^{(n)},\ldots,x_m^{(n-2)},x_m^{(n-1)}) = lim_{m\to\infty}gx_m^{(n)} = x^{(n)},
\end{aligned}$$

for some $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)} \in X$ are satisfied.

The main aim of this paper is to introduce compatiblity for n-tuples and prove n-tupled coincidence point results for pair if compatible maps as well as n-tupled fixed points results in partially ordered complete G-metric spaces satisfying weakly-contractive type condition enjoying mixed monotone property. Basically our theorems generalizes the Banach and Kannan contraction condition, respectively.

3 Main Results

Theorem 3.1. Let $(X,G,\preceq)$ be partially ordered complete G-metric space. Let $F: X^n \to X$ and $g: X \to X$ be two mappings such that F has mixed g-monotone property on X and satisfies the following conditions:

$$\begin{aligned}
&G(F(x^{(1)},x^{(2)},x^{(3)},...,x^{(n)}),F(y^{(1)},y^{(2)},y^{(3)},...,y^{(n)}),F(z^{(1)},z^{(2)},z^{(3)},...,z^{(n)}))\\
&\leq \{\frac{G(gx^{(1)},gy^{(1)},gz^{(1)})+...+G(gx^{(n)},gy^{(n)},gz^{(n)})}{n}\} - \phi\{\frac{G(gx^{(1)},gy^{(1)},gz^{(1)})+...+G(gx^{(n)},gy^{(n)},gz^{(n)})}{n}\}, \quad (2)
\end{aligned}$$

for all $x^{(1)},x^{(2)},x^{(3)},...,x^{(n)},y^{(1)},y^{(2)},y^{(3)},...,y^{(n)},z^{(1)},z^{(2)},z^{(3)},...,z^{(n)} \in X$ with $gz^{(1)} \preceq gy^{(1)} \preceq gx^{(1)}$, $gz^{(2)} \succeq gy^{(2)} \succeq gx^{(2)}$,..., $gz^{(n)} \succeq gy^{(n)} \succeq gx^{(n)}$, where $gy^{(1)} \neq gz^{(1)}$, $gy^{(2)} \neq gz^{(2)}$, ..., $gy^{(n)} \neq gz^{(n)}$ and $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > 0$ for all $t \in (0,\infty)$. We assume the following hypothesis:

1. $F(X^n) \subseteq g(X)$, $g(X)$ is complete, g is continuous and F is g-compatible.
2. F is continuous or
3. (a) if a non-decreasing sequence $\{x_m\} \to x$, then $gx_m \preceq gx$, for all $m \geq 0$.
   (b) if a non-increasing sequence $\{y_m\} \to y$, then $gy \preceq gy_m$, for all $m \geq 0$.

and if there are $x_0^{(1)},x_0^{(2)},x_0^{(3)},...,x_0^{(n)} \in X$ such that $gx_0^{(1)} \preceq F(x_0^{(1)},x_0^{(2)},x_0^{(3)},...,x_0^{(n)})$, $gx_0^{(2)} \succeq F(x_0^{(2)},x_0^{(3)},...,x_0^{(n)},x_0^{(1)})$, $gx_0^{(3)} \preceq F(x_0^{(3)},...,x_0^{(n)},x_0^{(1)},x_0^{(2)}),\ldots,gx_0^{(n)} \succeq F(x_0^{(n)},x_0^{(1)},\ldots,x_0^{(n-2)},x_0^{(n-1)})$. Then F and g have n-tupled coincidence point in X.

Proof. Let $x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)} \in X$ such that

$$\left.\begin{array}{r} gx_0^{(1)} \preceq F(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)}) \\ gx_0^{(2)} \succeq F(x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)}, x_0^{(1)}) \\ gx_0^{(3)} \preceq F(x_0^{(3)}, \ldots, x_0^{(n)}, x_0^{(1)}, x_0^{(2)}) \\ \vdots \\ gx_0^{(n)} \succeq F(x_0^{(n)}, x_0^{(1)}, \ldots, x_0^{(n-1)}) \ \textit{when n is even.} \end{array}\right\} \tag{3}$$

Since $F(X^n) \subseteq g(X)$, We define, $x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, \ldots, x_1^{(n)} \in X$ such that

$$\left.\begin{array}{r} F(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)}) = gx_1^{(1)} \\ F(x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)}, x_0^{(1)}) = gx_1^{(2)} \\ F(x_0^{(3)}, \ldots, x_0^{(n)}, x_0^{(1)}, x_0^{(2)}) = gx_1^{(3)} \\ \vdots \\ F(x_0^{(n)}, x_0^{(1)}, \ldots, x_0^{(n-2)}, x_0^{(n-1)}) = gx_1^{(n)} \end{array}\right\} \tag{4}$$

Continuing the above procedure, we can construct m sequences $\{x_m^{(1)}\}$, $\{x_m^{(2)}\}$, $\{x_m^{(3)}\}$, ...and $\{x_m^{(n)}\}$ in X such that

$$\left.\begin{array}{r} F(x_m^{(1)}, x_m^{(2)}, x_m^{(3)}, \ldots, x_m^{(n)}) = gx_{m+1}^{(1)} \\ F(x_m^{(2)}, x_m^{(3)}, \ldots, x_m^{(n)}, x_m^{(1)}) = gx_{m+1}^{(2)} \\ F(x_m^{(3)}, \ldots, x_m^{(n)}, x_m^{(1)}, x_m^{(2)}) = gx_{m+1}^{(3)} \\ \vdots \\ F(x_m^{(n)}, x_m^{(1)}, \ldots\ldots\ldots, x_m^{(n-2)}, x_m^{(n-1)}) = gx_{m+1}^{(n)} \end{array}\right\} \tag{5}$$

We are going to divide the proof into several steps:

Step 1. We shall prove that for all $m \geq 0$,

$$gx_m^{(1)} \preceq gx_{m+1}^{(1)}, gx_m^{(2)} \succeq gx_{m+1}^{(2)}, gx_m^{(3)} \preceq gx_{m+1}^{(3)} \ldots gx_m^{(n)} \succeq gx_{m+1}^{(n)}. \tag{6}$$

By using (3.2) and (3.3),we have

$$\left.\begin{array}{r} gx_0^{(1)} \preceq F(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)}) = gx_1^{(1)} \\ gx_0^{(2)} \succeq F(x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)}, x_0^{(1)}) = gx_1^{(2)} \\ gx_0^{(3)} \preceq F(x_0^{(3)}, \ldots, x_0^{(n)}, x_0^{(1)}, x_0^{(2)}) = gx_1^{(3)} \\ \vdots \\ gx_0^{(n)} \succeq F(x_0^{(n)}, x_0^{(1)}, \ldots, x_0^{(n-2)}, x_0^{(n-1)}) = gx_1^{(n)}. \end{array}\right\} \tag{7}$$

So, (3.5) holds for $m = 0$. In the same way, using mixed monotone property of F we define,

$$\left.\begin{array}{r} F(x_1^{(1)},x_1^{(2)},x_1^{(3)},\ldots,x_1^{(n)}) = gx_2^{(1)} \\ F(x_1^{(2)},x_1^{(3)},\ldots,x_1^{(n)},x_1^{(1)}) = gx_2^{(2)} \\ F(x_1^{(3)},\ldots,x_1^{(n)},x_1^{(1)},x_1^{(2)}) = gx_2^{(3)} \\ \vdots \\ F(x_1^{(n)},x_1^{(1)},\ldots,x_1^{(n-2)},x_1^{(n-1)}) = gx_2^{(n)} \end{array}\right\} \tag{8}$$

Then

$$\begin{aligned} & gx_2^{(1)} = F(x_1^{(1)},x_1^{(2)},x_1^{(3)},\ldots,x_1^{(n)}) \succeq F(x_0^{(1)},x_1^{(2)},x_1^{(3)},\ldots,x_1^{(n)}) \succeq F(x_0^{(1)},x_0^{(2)},x_1^{(3)},\ldots,x_1^{(n)}) \\ & \succeq F(x_0^{(1)},x_0^{(2)},x_0^{(3)},\ldots,x_0^{(n)}) = gx_1^{(1)}. \\ & gx_2^{(2)} = F(x_1^{(2)},x_1^{(3)},x_1^{(4)},\ldots,x_1^{(n)},x_1^{(1)}) \preceq F(x_1^{(2)},x_1^{(3)},x_1^{(4)},\ldots,x_1^{(n)},x_0^{(1)}) \preceq F(x_1^{(2)},x_1^{(3)},x_1^{(4)},.. \\ & \ldots,x_0^{(n)},x_0^{(1)}) \preceq \ldots \preceq F(x_0^{(2)},x_0^{(3)},x_0^{(4)},\ldots,x_0^{(1)}) = gx_1^{(2)}. \end{aligned}$$

Similarly,

$$gx_2^{(3)} \succeq gx_1^{(3)}, gx_2^{(4)} \preceq gx_1^{(4)}, \ldots, gx_2^{(n)} \preceq gx_1^{(n)}.$$

Suppose that (3.5) holds for some $m > 0$, as F has the g-mixed monotone property, we have from (3.4) that In the same way, using g-mixed monotone property of F we define,

$$\begin{aligned} gx_{m+1}^{(1)} = F(x_m^{(1)},x_m^{(2)},x_m^{(3)},\ldots,x_m^{(n)}) &\preceq F(x_{m+1}^{(1)},x_m^{(2)},x_m^{(3)},\ldots,x_m^{(n)}) \\ &\preceq F(x_{m+1}^{(1)},x_{m+1}^{(2)},x_m^{(3)},\ldots,x_m^{(n)}) \\ &\preceq F(x_{m+1}^{(1)},x_{m+1}^{(2)},x_{m+1}^{(3)},\ldots,x_{m+1}^{(n)}) \\ &= gx_{m+2}^{(1)}. \end{aligned}$$

$$\begin{aligned} gx_{m+1}^{(2)} = F(x_m^{(2)},x_m^{(3)},\ldots,x_m^{(n)},x_m^{(1)}) &\succeq F(x_{m+1}^{(2)},x_m^{(3)},\ldots,x_m^{(n)},x_m^{(1)}) \\ &\succeq F(x_{m+1}^{(2)},x_{m+1}^{(3)},\ldots,x_m^{(n)},x_m^{(1)}) \\ &\succeq F(x_{m+1}^{(2)},x_{m+1}^{(3)},\ldots,x_{m+1}^{(n)},x_m^{(1)}) \\ &\succeq F(x_{m+1}^{(2)},x_{m+1}^{(3)},\ldots,x_{m+1}^{(n)},x_{m=1}^{(1)}) \end{aligned}$$

$$= gx_{m+2}^{(2)}.$$

Thus by mathematical induction we conclude that (3.5) holds for all $m \geq 0$. Therefore,

$$\left.\begin{array}{r} gx_0^{(1)} \preceq gx_1^{(1)} \preceq gx_2^{(1)} \preceq \ldots \preceq gx_m^{(1)} \preceq gx_{m+1}^{(1)} \preceq \cdots \\ gx_0^{(2)} \succeq gx_1^{(2)} \succeq gx_2^{(2)} \succeq \ldots \succeq gx_m^{(2)} \succeq gx_{m+1}^{(2)} \succeq \cdots \\ gx_0^{(3)} \preceq gx_1^{(3)} \preceq gx_2^{(3)} \preceq \ldots \preceq gx_m^{(3)} \preceq gx_{m+1}^{(3)} \preceq \cdots \\ \vdots \\ gx_0^{(n)} \succeq gx_1^{(n)} \succeq gx_2^{(n)} \succeq \ldots \succeq gx_m^{(n)} \succeq gx_{m+1}^{(n)} \succeq \ldots. \end{array}\right\} \tag{9}$$

This completes the proof of our claim.

Step 2. Now we shall show that

$$lim_{m\to\infty}\{G(gx_m^{(1)}, gx_{m+1}^{(1)}, gx_{m+1}^{(1)}) + G(gx_m^{(2)}, gx_{m+1}^{(2)}, gx_{m+1}^{(2)}) + \ldots + G(gx_m^{(n)}, gx_{m+1}^{(n)}, gx_{m+1}^{(n)})\} = 0. \tag{10}$$

From the (3.4), we have

$$\begin{aligned} A_m^1 = G(gx_m^{(1)}, gx_{m+1}^{(1)}, gx_{m+1}^{(1)}) &= G(F(x_{m-1}^{(1)}, x_{m-1}^{(2)}, \ldots, x_{m-1}^{(n)}), F(x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(n)}), F(x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(n)})) \\ &\leq \{\tfrac{G(gx_{m-1}^{(1)}, gx_m^{(1)}, gx_m^{(1)}) + \ldots + G(gx_{m-1}^{(n-1)}, gx_m^{(n-1)}, gx_m^{(n-1)}) + G(gx_{m-1}^{(n)}, gx_m^{(n)}, gx_m^{(n)})}{n}\} - \\ &\quad \phi\{\tfrac{G(gx_{m-1}^{(1)}, gx_m^{(1)}, gx_m^{(1)}) + \ldots + G(gx_{m-1}^{(n-1)}, gx_m^{(n-1)}, gx_m^{(n-1)}) + G(gx_{m-1}^{(n)}, gx_m^{(n)}, gx_m^{(n)})}{n}\} \end{aligned}$$

$$\begin{aligned} A_m^2 = G(gx_m^{(2)}, gx_{m+1}^{(2)}, gx_{m+1}^{(2)}) &= G(F(x_{m-1}^{(2)}, \ldots, x_{m-1}^{(n)}, x_{m-1}^{(1)}), F(x_m^{(2)}, \ldots, x_m^{(n)}, x_m^{(1)}), F(x_m^{(2)}, \ldots, x_m^{(n)}, x_m^{(1)})) \\ &\leq \{\tfrac{G(gx_{m-1}^{(2)}, gx_m^{(2)}, gx_m^{(2)}) + \ldots + G(gx_{m-1}^{(n)}, gx_m^{(n)}, gx_m^{(n)}) + G(gx_{m-1}^{(1)}, gx_m^{(1)}, gx_m^{(1)})}{n}\} - \\ &\quad \phi\{\tfrac{G(gx_{m-1}^{(2)}, gx_m^{(2)}, gx_m^{(2)}) + \ldots + G(gx_{m-1}^{(n)}, gx_m^{(n)}, gx_m^{(n)}) + G(gx_{m-1}^{(1)}, gx_m^{(1)}, gx_m^{(1)})}{n}\} \\ &\quad \vdots \end{aligned}$$

Similarly, we can including write

$$\begin{aligned} A_m^n = G(gx_m^{(n)}, gx_{m+1}^{(n)}, gx_{m+1}^{(n)}) &= G(F(x_{m-1}^{(n)}, x_{m-1}^{(1)}, \ldots, x_{m-1}^{(n-1)}), F(x_m^{(n)}, x_m^{(1)}, \ldots, x_m^{(n-1)}), F(x_m^{(n)}, x_m^{(1)}, \ldots, x_m^{(n-1)})) \\ &\leq \{\tfrac{G(gx_{m-1}^{(n)}, gx_m^{(n)}, gx_m^{(n)}) + G(gx_{m-1}^{(1)}, gx_m^{(1)}, gx_m^{(1)}) + \ldots + G(gx_{m-1}^{(n-1)}, gx_m^{(n-1)}, gx_m^{(n-1)})}{n}\} - \\ &\quad \phi\{\tfrac{G(gx_{m-1}^{(n)}, gx_m^{(n)}, gx_m^{(n)}) + G(gx_{m-1}^{(1)}, gx_m^{(1)}, gx_m^{(1)}) + \ldots + G(gx_{m-1}^{(n-1)}, gx_m^{(n-1)}, gx_m^{(n-1)})}{n}\}. \end{aligned}$$

Let for all, $m \geq 0$, adding the above inequality, we obtain

$$\begin{aligned} \delta_{m+1} &= G(gx_m^{(1)}, gx_{m+1}^{(1)}, gx_{m+1}^{(1)}) + G(gx_m^{(2)}, gx_{m+1}^{(2)}, gx_{m+1}^{(2)}) + \ldots + G(gx_m^{(n)}, gx_{m+1}^{(n)}, gx_{m+1}^{(n)}) \\ \delta_{m+1} &\leq \{G(gx_{m-1}^{(n)}, gx_m^{(n)}, gx_m^{(n)}) + G(gx_{m-1}^{(1)}, gx_m^{(1)}, gx_m^{(1)}) + \ldots + G(gx_{m-1}^{(n-1)}, gx_m^{(n-1)}, gx_m^{(n-1)})\} - \end{aligned}$$

$$n\phi\{\frac{G(gx^{(n)}_{m-1},gx^{(n)}_{m},gx^{(n)}_{m})+G(gx^{(1)}_{m-1},gx^{(1)}_{m},gx^{(1)}_{m})+\ldots+G(gx^{(n-1)}_{m-1},gx^{(n-1)}_{m},gx^{(n-1)}_{m})}{n}\}$$

$$\delta_{m+1} \leq \delta_m - n\phi\{\frac{\delta_m}{n}\} \tag{11}$$

$$\delta_{m+1} \leq \delta_m. \qquad \text{(by the property of } \phi\text{-function)}$$

Therefore the sequence $\{\delta_m\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exist $\delta \geq 0$ such that $lim_{n\to\infty}\delta_m = \delta$. Assume $\delta > 0$. Then from (3.10) again, taking limit $m \to \infty$, we get

$$lim_{m\to\infty}\delta_{m+1} \leq lim_{m\to\infty}\delta_m - n\phi\{lim_{m\to\infty}\frac{\delta_m}{n}\}$$
$$\Rightarrow \delta \leq \delta - n\phi\{\tfrac{\delta}{n}\},$$

which is a contradiction. hence

$$lim_{m\to\infty}\delta_m = 0. \tag{12}$$

$$\Rightarrow lim_{m\to\infty}\{G(gx^{(1)}_m, gx^{(1)}_{m+1}, gx^{(1)}_{m+1}) + G(gx^{(2)}_m, gx^{(2)}_{m+1}, gx^{(2)}_{m+1}) + \ldots + G(gx^{(n)}_m, gx^{(n)}_{m+1}, gx^{(n)}_{m+1})\} = 0.$$

Hence

$$\left.\begin{aligned} lim_{m\to\infty}\{G(gx^{(1)}_m, gx^{(1)}_{m+1}, gx^{(1)}_{m+1})\} = 0 \\ lim_{m\to\infty}\{G(gx^{(2)}_m, gx^{(2)}_{m+1}, gx^{(2)}_{m+1})\} = 0 \\ \vdots \\ lim_{m\to\infty}\{G(gx^{(n)}_m, gx^{(n)}_{m+1}, gx^{(n)}_{m+1})\} = 0. \end{aligned}\right\} \tag{13}$$

This prove our claim.

Step 3. Next we have to show that $\{gx^{(1)}_m\}$, $\{gx^{(2)}_m\}$, $\{gx^{(3)}_m\}$, ... and $\{gx^{(n)}_m\}$ are Cauchy sequence. If possible, let at least one of $\{gx^{(1)}_m\}$, ... , $\{gx^{(n)}_m\}$ be not a Cauchy sequence. Then there exist an $\varepsilon > 0$ and sequence of natural numbers $\{n(k)\}$ and $\{l(k)\}$ for which $n(k) > l(k) \geq k$, and such that for all $k \geq 1$, either Hence

$$\left.\begin{aligned} G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) \geq \varepsilon \ \ or \\ G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k)}, gx^{(2)}_{n(k)}) \geq \varepsilon \ \ or \\ G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)}) \geq \varepsilon \ \ or \\ \vdots \\ G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)}) \geq \varepsilon. \end{aligned}\right\} \tag{14}$$

Then for all $k \geq 1$,

$$g_k = G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k)}, gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)}) \geq \varepsilon. \tag{15}$$

Now, corresponding to $l(k)$ we can choose $n(k)$ to be the smallest positive integer for which (3.14) holds. Then

$$G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)-1}, gx^{(1)}_{n(k-1)}) + G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k)-1}, gx^{(2)}_{n(k)-1}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)-1}, gx^{(n)}_{n(k)-1}) < \varepsilon. \tag{16}$$

Further, from (3.14) and (3.15), for all $k \geq 1$, we have

$$\varepsilon \leq g_k = G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + G(gx^{(2)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)}).$$

By triangle inequality, we have

$$\leq G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k-1)}, gx^{(1)}_{n(k)-1}) + G(gx^{(1)}_{n(k)-1}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k-1)}, gx^{(2)}_{n(k)-1}) +$$

$$G(gx^{(2)}_{n(k)-1}, gx^{(2)}_{n(k)}, gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)-1}, gx^{(n)}_{n(k)-1}) + G(gx^{(n)-1}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)})$$

$$\leq G(x^{(1)}_{l(k)}, x^{(1)}_{n(k-1)}, x^{(1)}_{n(k)-1}) + G(x^{(2)}_{l(k)}, x^{(2)}_{n(k)-1}, x^{(2)}_{n(k)-1}) + \ldots + G(x^{(n)}_{l(k)}, gx^{(n)}_{n(k)-1}, gx^{(n)}_{n(k)-1}) + A^{1}_{n(k)-1}$$

$$+ A^{2}_{n(k)-1} + \ldots + A^{n}_{n(k)-1}$$

$$< \varepsilon + A^{1}_{n(k)-1} + A^{2}_{n(k)-1} + \ldots + A^{n}_{n(k)-1}.$$

Taking limit $k \to \infty$, we have

$$lim_{k\to\infty} g_k = \varepsilon. \tag{17}$$

For all $k \geq 1$, we obtain,

$$G(gx^{(1)}_{l(k)+1}, gx^{(1)}_{n(k+1)}, gx^{(1)}_{n(k)+1}) = G(F(x^{(1)}_{l(k)}, x^{(2)}_{l(k)}, \ldots, x^{(n)}_{l(k)}), F(x^{(1)}_{n(k)}, x^{(2)}_{n(k)}), \ldots, x^{(n)}_{n(k)}),$$

$$F(x^{(1)}_{n(k)}, x^{(2)}_{n(k)}), \ldots, x^{(n)}_{n(k)})) \leq \{\frac{G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k)}, gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)})}{n}\} -$$

$$\phi\{\frac{G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k)}, gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)})}{n}\}$$

$$G(gx^{(2)}_{l(k)+1}, gx^{(2)}_{n(k+1)}, gx^{(2)}_{n(k)+1}) = G(F(x^{(2)}_{l(k)}, \ldots, x^{(n)}_{l(k)}, x^{(1)}_{l(k)}), F(x^{(2)}_{n(k)}, \ldots, x^{(n)}_{n(k)}), x^{(1)}_{n(k)}),$$

$$F(x^{(2)}_{n(k)}, \ldots, x^{(n)}_{n(k)}, x^{(1)}_{n(k)}) \leq \{\frac{G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k)}, gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)}) + G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)})}{n}\} -$$

$$\phi\{\frac{G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k)}, gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)} + G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)})}{n}\}$$

$$\vdots$$

Similarly, we have

$$G(gx^{(n)}_{l(k)+1}, gx^{(n)}_{n(k+1)}, gx^{(n)}_{n(k)+1}) = G(F(x^{(n)}_{l(k)}, x^{(1)}_{l(k)}, \ldots, x^{(n-1)}_{l(k)}), F(x^{(n)}_{n(k)}, x^{(1)}_{n(k)}), \ldots, x^{(n-1)}_{n(k)}),$$

$$F(x^{(n)}_{n(k)}, x^{(1)}_{n(k)}), \ldots, x^{(n-1)}_{n(k)})) \leq \{\frac{G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)}) + G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + \ldots + G(gx^{(n-1)}_{l(k)}, gx^{(n-1)}_{n(k)}, gx^{(n-1)}_{n(k)})}{n}\} -$$

$$\phi\{\frac{G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)}) + G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + \ldots + G(gx^{(n-1)}_{l(k)}, gx^{(n-1)}_{n(k)}, gx^{(n-1)}_{n(k)})}{n}\}.$$

Now adding above inequalities, we get

$$G(gx^{(1)}_{l(k)+1}, gx^{(1)}_{n(k)+1}, gx^{(1)}_{n(k)+1}) + G(gx^{(2)}_{l(k)+1}, gx^{(2)}_{n(k)+1}, gx^{(2)}_{n(k)+1}) + \ldots + G(gx^{(n)}_{l(k)+1}, gx^{(n)}_{n(k+1)}, gx^{(n)}_{n(k)+1})$$

$$\leq n\{\frac{G(gx^{(1)}_{l(k)}, gx^{(1)}_{n(k)}, gx^{(1)}_{n(k)}) + G(gx^{(2)}_{l(k)}, gx^{(2)}_{n(k)}, gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)}, gx^{(n)}_{n(k)}, gx^{(n)}_{n(k)})}{n}\} -$$

$$n\phi\{\frac{G(gx^{(1)}_{l(k)},gx^{(1)}_{n(k)},gx^{(1)}_{n(k)})+G(gx^{(2)}_{l(k)},gx^{(2)}_{n(k)},gx^{(2)}_{n(k)})+\ldots+G(gx^{(n)}_{l(k)},gx^{(n)}_{n(k)},gx^{(n)}_{n(k)})}{n}\}.$$

Again, for all $k \geq 1$ and by using the triangle inequality, we have

$$\begin{aligned}
g_k = G(gx^{(1)}_{l(k)},gx^{(1)}_{n(k)},gx^{(1)}_{n(k)}) + G(gx^{(2)}_{l(k)},gx^{(2)}_{n(k)},gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)},gx^{(n)}_{n(k)},gx^{(n)}_{n(k)})\\
&\leq G(gx^{(1)}_{l(k)},gx^{(1)}_{l(k)+1},gx^{(1)}_{l(k)+1}) + G(gx^{(1)}_{l(k)+1},gx^{(1)}_{n(k)},gx^{(1)}_{n(k)}) + G(gx^{(2)}_{l(k)},gx^{(2)}_{l(k)+1},gx^{(2)}_{l(k)+1})\\
&+G(gx^{(2)}_{l(k)+1},gx^{(2)}_{n(k)},gx^{(2)}_{n(k)}) + \ldots + G(gx^{(n)}_{l(k)},gx^{(n)}_{l(k)+1},gx^{(n)}_{l(k)+1}) + G(gx^{(n)}_{l(k)+1},gx^{(n)}_{n(k)},gx^{(n)}_{n(k)})\\
&\leq G(x^{(1)}_{l(k)},gx^{(1)}_{l(k)+1},gx^{(1)}_{l(k)+1}) + G(gx^{(1)}_{l(k)+1},gx^{(1)}_{n(k)+1},gx^{(1)}_{n(k)+1}) + G(gx^{(1)}_{n(k)+1},gx^{(1)}_{n(k)},gx^{(1)}_{n(k)})\\
&+\ldots + G(gx^{(n)}_{l(k)},gx^{(n)}_{l(k)+1},gx^{(n)}_{l(k)+1}) + G(gx^{(n)}_{l(k)+1},gx^{(n)}_{n(k)+1},gx^{(n)}_{n(k)+1}) + G(gx^{(n)}_{l(k)+1},gx^{(n)}_{n(k)},gx^{(n)}_{n(k)}).
\end{aligned}$$

Taking limit $k \to \infty$, we have

$$\varepsilon = lim_{k\to\infty} g_k \leq lim_{k\to\infty}(G(gx^{(1)}_{l(k)+1},gx^{(1)}_{n(k+1)},gx^{(1)}_{n(k)+1}) + \ldots + G(gx^{(n)}_{l(k)+1},gx^{(n)}_{n(k)+1},gx^{(n)}_{n(k)+1})). \tag{18}$$

Now again,

$$\begin{aligned}
G(gx^{(1)}_{l(k)+1},gx^{(1)}_{n(k)+1},gx^{(1)}_{n(k)+1}) + G(gx^{(2)}_{l(k)+1},gx^{(2)}_{n(k)+1},gx^{(2)}_{n(k)+1}) + \ldots + G(gx^{(n)}_{l(k)+1},gx^{(n)}_{n(k)+1},gx^{(n)}_{n(k)+1}) \leq\\
G(gx^{(1)}_{l(k)+1},gx^{(1)}_{l(k)},gx^{(1)}_{l(k)}) + G(gx^{(1)}_{l(k)},gx^{(1)}_{n(k)},gx^{(1)}_{n(k)}) + G(gx^{(1)}_{n(k)},gx^{(1)}_{n(k)+1},gx^{(1)}_{n(k)+1}) + \ldots\\
+G(gx^{(n)}_{l(k)+1},gx^{(n)}_{l(k)},gx^{(n)}_{l(k)}) + G(gx^{(n)}_{l(k)},gx^{(n)}_{n(k)},gx^{(n)}_{n(k)}) + G(gx^{(n)}_{n(k)},gx^{(n)}_{n(k)+1},gx^{(n)}_{n(k)+1}).
\end{aligned}$$

Taking limit $k \to \infty$, we have

$$lim_{k\to\infty}(G(gx^{(1)}_{l(k)+1},gx^{(1)}_{n(k)+1},gx^{(1)}_{n(k)+1}) + \ldots + G(gx^{(n)}_{l(k)+1},gx^{(n)}_{n(k)+1},gx^{(n)}_{n(k)+1})) \leq lim_{k\to\infty} g_k = \varepsilon. \tag{19}$$

By (3.17) and (3.18), we get

$$lim_{k\to\infty}(G(gx^{(1)}_{l(k)+1},gx^{(1)}_{n(k+1)},gx^{(1)}_{n(k)+1}) + \ldots + G(gx^{(n)}_{l(k)+1},gx^{(n)}_{n(k+1)},gx^{(n)}_{n(k)+1})) = \varepsilon. \tag{20}$$

By (3.19), we have

$$\varepsilon \leq \varepsilon - n\phi\{\tfrac{\varepsilon}{n}\},$$

which is a contradiction. Therefore $\{gx^{(1)}_m\}$, $\{gx^{(2)}_m\}$, $\{gx^{(3)}_m\}$, $\ldots$ and $\{gx^{(n)}_m\}$ are cauchy sequences in X. From the completeness of X there exist $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in X$ such that

$$\left.\begin{array}{rl}
lim_{n\to\infty}F(x^{(1)}_m,x^{(2)}_m,\ldots,x^{(n)}_m) = lim_{n\to\infty}gx^{(1)}_m \to x^{(1)} as & m\to\infty\\
lim_{n\to\infty}F(x^{(2)}_m,\ldots,x^{(n)}_m,x^{(1)}_m) = lim_{n\to\infty}gx^{(2)}_m \to x^{(2)} as & m\to\infty\\
lim_{n\to\infty}F(x^{(3)}_m,\ldots,x^{(n)}_m,x^{(1)}_m,x^{(2)}_m) = lim_{n\to\infty}gx^{(3)}_m \to x^{(3)} as & m\to\infty\\
\vdots & \\
lim_{n\to\infty}F(x^{(n)}_m,x^{(1)}_m,\ldots,x^{(n-1)}_m) = lim_{n\to\infty}gx^{(n)}_m \to x^{(n)} as & m\to\infty.
\end{array}\right\} \tag{21}$$

Step 4. Now we have to show that F has n-tupled fixed points. For this, Let the condition (1) of the theorem holds. that is F is continuous. Since F is g-compatible, we have from (3.20)

$$\left.\begin{array}{r} lim_{m\to\infty}G(g(F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}),F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}))=0 \\ lim_{m\to\infty}G(g(F(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)}),F(gx_m^{(2)},\ldots,gx_m^{(n)},gx_m^{(1)}),F(gx_m^{(2)},\ldots,gx_m^{(n)},gx_m^{(1)}))=0 \\ \vdots \\ lim_{m\to\infty}G(g(F(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}),F(gx_m^{(n)},gx_m^{(1)},\ldots,gx_m^{(n-1)}),F(gx_m^{(n)},gx_m^{(1)},\ldots,gx_m^{(n-1)}))=0 \end{array}\right\} \tag{22}$$

Then for all $m\geq 0$, we have

$$G(gx^{(1)},F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}),F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}))\leq G(gx^{(1)},g(F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)})),$$

$$g(F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)})))+G(g(F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)})),F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}))).$$

Taking $m\to\infty$ in above inequality using (3.20), (3.21) and continuous of F and g, we have

$$\begin{array}{r} lim_{m\to\infty}G(gx^{(1)},F(x^{(1)},x^{(2)},\ldots,x^{(n)}),F(x^{(1)},x^{(2)},\ldots,x^{(n)}))=0 \text{ that is } gx^{(1)}=F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}) \\ lim_{m\to\infty}G(gx^{(2)},F(x^{(2)},\ldots,x^{(n)},x^{(1)}),F(x^{(1)},\ldots,x^{(n)},x^{(1)}))=0 \text{ that is } gx^{(2)}=F(x^{(2)},\ldots,x^{(n)},x^{(1)}) \\ \vdots \\ lim_{m\to\infty}G(gx^{(n)},F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}))=0 \text{ that is } gx^{(n)}=F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}). \end{array}$$

Hence the element $x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}\in X^n$ is n-tupled coincidence point of mapping $F:X^n\to X$ and $g:X\to X$.

Next, Let condition (2) holds. by (3.8) and (3.20), we have

$$ggx_m^{(1)}\preceq gx^{(1)},gx^{(2)}\preceq ggx_m^{(2)},ggx_m^{(3)}\preceq gx^{(3)},\ldots,gx^{(n)}\preceq ggx_m^{(n)}.$$

Since we have F and g is g-compatible and g is continuous by (3.20) and (3.21) we have

$$\left.\begin{array}{r} lim_{n\to\infty}ggx_m^{(1)}=gx^{(1)}=lim_{n\to\infty}g(F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}))=lim_{n\to\infty}F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}) \\ lim_{n\to\infty}ggx_m^{(1)}=gx^{(1)}=lim_{n\to\infty}g(F(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)}))=lim_{n\to\infty}F(gx_m^{(n)},\ldots,gx_m^{(1)},gx_m^{(n-1)}) \\ \vdots \\ lim_{n\to\infty}ggx_m^{(n)}=gx^{(n)}=lim_{n\to\infty}g(F(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}))=lim_{n\to\infty}F(gx_m^{(n)},gx_m^{(1)},\ldots,gx_m^{(n-1)}) \end{array}\right\} \tag{23}$$

Now using triangle inequality, we have

$$G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),gx^{(1)},gx^{(1)})\leq G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),ggx_{m+1}^{(1)},ggx_{m+1}^{(1)})+G(ggx_{m+1}^{(1)},gx^{(1)},gx^{(1)})$$

that is,

$$G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),gx^{(1)},gx^{(1)})\leq G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),gF(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),$$
$$gF(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}))+G(ggx_{m+1}^{(1)},gx^{(1)},gx^{(1)})$$

Taking $m\to\infty$ in the theorem in the above inequality, using (3.22) we have

$$lim_{n\to\infty}G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),gx^{(1)},gx^{(1)}) \le lim_{n\to\infty}G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),gF(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),$$
$$gF(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}))+lim_{n\to\infty}G(ggx_{m+1}^{(1)},gx^{(1)},gx^{(1)})$$

$$lim_{n\to\infty}G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),gx^{(1)},gx^{(1)}) \le lim_{n\to\infty}G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}),$$
$$F(gx_m^{(1)},gx_m^{(2)},\ldots,gx_m^{(n)}))$$
$$\le lim_{n\to\infty}\{\tfrac{G(gx^{(1)},ggx_m^{(1)},ggx_m^{(1)})+\ldots+G(gx^{(n)},ggx_m^{(n)},ggx_m^{(n)})}{n}\}-$$
$$lim_{n\to\infty}\phi\{\tfrac{G(gx^{(1)},ggx_m^{(1)},ggx_m^{(1)})+\ldots+G(gx^{(n)},ggx_m^{(n)},ggx_m^{(n)})}{n}\}.$$

Using (3.22), Lemama 2.4 and property of ϕ-function, we have

$$G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),gx^{(1)},gx^{(1)})=0 \text{ that is } F(x^{(1)},x^{(2)},\ldots,x^{(n)})=gx^{(1)}$$

Again we have,

$$G(F(x^{(2)},\ldots,x^{(n)},x^{(1)}),gx^{(2)},gx^{(2)})=0 \text{ that is } F(x^{(2)},\ldots,x^{(n)},x^{(1)})=gx^{(2)}$$
$$\vdots$$

Similarly, we can have

$$G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),gx^{(n)},gx^{(n)}) \le G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),ggx_{m+1}^{(n)},ggx_{m+1}^{(n)})+G(ggx_{m+1}^{(n)},gx^{(n)},gx^{(n)})$$

that is,

$$G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),gx^{(n)},gx^{(n)}) \le G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),gF(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}),$$
$$gF(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}))+G(ggx_{m+1}^{(n)},gx^{(n)},gx^{(n)}).$$

Taking $m\to\infty$ in the theorem in the above inequality, using (3.22) we have

$$lim_{n\to\infty}G(F(x^{(n)},x^{(1)},\ldots,x^{(n1)}),gx^{(n)},gx^{(n)}) \le lim_{n\to\infty}G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),gF(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}),$$
$$gF(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}))+lim_{n\to\infty}G(ggx_{m+1}^{(n)},gx^{(n)},gx^{(n)})$$

$$lim_{n\to\infty}G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),gx^{(n)},gx^{(n)}) \le lim_{n\to\infty}G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),F(gx_m^{(n)},gx_m^{(n)},\ldots,gx_m^{(n)}),$$
$$F(gx_m^{(n)},gx_m^{(1)},\ldots,gx_m^{(n-1)}))$$
$$\le lim_{n\to\infty}\{\tfrac{G(gx^{(n)},ggx_m^{(n)},ggx_m^{(n)})+\ldots+G(gx^{(n-1)},ggx_m^{(n-1)},ggx_m^{(n-1)})}{n}\}-$$
$$lim_{n\to\infty}\phi\{\tfrac{G(gx^{(n)},ggx_m^{(n)},ggx_m^{(n)})+\ldots+G(gx^{(n-1)},ggx_m^{(n-1)},ggx_m^{(n-1)})}{n}\}.$$

Using (3.22), Lemma 2.4 and property of ϕ-function, we have

$$G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),gx^{(n)},gx^{(n)})=0 \text{ that is } F(x^{(n)},x^{(1)},\ldots,x^{(n-1)})=gx^{(n)}.$$

Hence the element $x^{(1)},x^{(2)},\ldots,x^{(n)}\in X^n$ is n-tupled coincidence point of the mapping $F:X^n\to X$ and $g:X\to X$. This completes the proof of the theorem.

Theorem 3.2. Let $(X,G,\preceq)$ be partially ordered complete G-metric space. Let $F: X^n \to X$ be a mapping such that F has mixed monotone property and satisfies the following conditions:

$$\begin{aligned}&G(F(x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)}),F(y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)}),F(y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)})) \leq \\ &\{M(x^{(1)},x^{(2)},\ldots,x^{(n)},y^{(1)},y^{(2)},\ldots,y^{(n)})\} - \phi\{M(x^{(1)},x^{(2)},\ldots,x^{(n)},y^{(1)},y^{(2)},\ldots,y^{(n)})\}\end{aligned} \tag{24}$$

where

$$\begin{aligned}&M(x^{(1)},x^{(2)},\ldots,x^{(n)},y^{(1)},y^{(2)},\ldots,y^{(n)}) = \frac{1}{2n}\{G(x^{(1)},F(x^{(1)},x^{(2)},\ldots,x^{(n)}),F(x^{(1)},x^{(2)},\ldots,x^{(n)}))\\ &+G(x^{(2)},F(x^{(2)},\ldots,x^{(n)},x^{(1)}),F(x^{(2)},\ldots,x^{(n)},x^{(1)}))+\ldots+G(x^{(n)},F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),F(x^{(n)},x^{(1)}\\ &,\ldots,x^{(n-1)}))+G(y^{(1)},F(y^{(1)},y^{(2)},\ldots,y^{(n)}),F(y^{(1)},y^{(2)},\ldots,y^{(n)}))+G(y^{(2)},F(y^{(2)},\ldots,y^{(n)},y^{(1)}),F(y^{(2)},\\ &\ldots,y^{(n)},y^{(1)}))+\ldots+G(y^{(n)},F(y^{(n)},y^{(1)},\ldots,y^{(n-1)}),F(y^{(n)},y^{(1)},\ldots,y^{(n-1)}))\}\end{aligned}$$

for all $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)} \in X$ and $y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)} \in X$ with $y^{(1)} \preceq x^{(1)}$, $x^{(2)} \preceq y^{(2)} \ldots x^{(n)} \preceq y^{(n)}$. and $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > 0$ for all $t \in (0,\infty)$. Also suppose that

1. F is continuous or
2. (a) if a non-decreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$, for all m.
   (b) if a non-increasing sequence $\{y_m\} \to y$, then $y \preceq y_m$, for all m.

and if there are $x_0^{(1)},x_0^{(2)},x_0^{(3)},\ldots,x_0^{(n)} \in X$ such that $x_0^{(1)} \preceq F(x_0^{(1)},x_0^{(2)},x_0^{(3)},...,x_0^{(n)})$, $x_0^{(2)} \succeq F(x_0^{(2)},x_0^{(3)},...,x_0^{(n)},x_0^{(1)})$, $x_0^{(3)} \preceq F(x_0^{(3)},...,x_0^{(n)},x_0^{(1)},x_0^{(2)})$,...,$x_0^{(n)} \succeq F(x_0^{(n)},x_0^{(1)},\ldots,x_0^{(n-2)},x_0^{(n-1)})$ when n is even. Then there exist $x^{(1)},x^{(2)},x^{(3)},...,x^{(n)}) \in X$ such that

$$\begin{aligned}F(x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)}) &= x^{(1)}\\ F(x^{(2)},x^{(3)},\ldots,x^{(n)},x^{(1)}) &= x^{(2)}\\ F(x^{(3)},\ldots,x^{(n)},x^{(1)},x^{(2)}) &= x^{(3)}\\ &\vdots\\ F(x^{(n)},x^{(1)},\ldots,x^{(n-2)},x^{(n-1)}) &= x^{(n)}\end{aligned}$$

that is, F has n-tupled fixed point in X.

Proof. Step 1 is same as the Theorem 3.1. Proof of this theorem is differ from the Step 2.

Step 2:

$$\begin{aligned}&G(x_m^{(1)},x_{m+1}^{(1)},x_{m+1}^{(1)}) = G(F(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)}),F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}))\\ &\leq \{M(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_m^{(1)},\ldots,x_m^{(n)})\} - \phi\{M(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_m^{(1)},\ldots,x_m^{(n)})\}\\ &G(x_m^{(2)},x_{m+1}^{(2)},x_{m+1}^{(2)}) = G(F(x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_{m-1}^{(1)}),F(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)}),F(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)}))\\ &\leq \{M(x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_{m-1}^{(1)},x_m^{(2)},\ldots,x_m^{(1)})\} - \phi\{M(x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_{m-1}^{(1)},x_m^{(2)},\ldots,x_m^{(1)}))\}\\ &\vdots\end{aligned}$$

Now again,

$$\begin{aligned}&G(x_m^{(n)},x_{m+1}^{(n)},x_{m+1}^{(n)}) = G(F(x_{m-1}^{(n)},,x_{m-1}^{(1)},\ldots,x_{m-1}^{(n-1)}),F(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}),F(x_m^{(n)},,x_m^{(1)},\ldots,x_m^{(n-1)}))\\ &\leq \{M(x_{m-1}^{(n)},x_{m-1}^{(1)},\ldots,x_{m-1}^{(n-1)},x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)})\} - \phi\{M(x_{m-1}^{(n)},,x_{m-1}^{(1)},\ldots,x_{m-1}^{(n-1)},x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}))\}\end{aligned}$$

where

$$M(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_m^{(1)},\ldots,x_m^{(n)}) = \frac{1}{2n}\{G(x_{m-1}^{(1)},F(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)}),$$
$$F(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)})+G(x_{m-1}^{(2)},F(x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_{m-1}^{(1)}),F(x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_{m-1}^{(1)})$$
$$+\ldots+G(x_{m-1}^{(n)},F(x_{m-1}^{(n)},x_{m-1}^{(1)},\ldots,x_{m-1}^{(n-1)}),F(x_{m-1}^{(n)},x_{m-1}^{(1)},\ldots,x_{m-1}^{(n-1)}))$$
$$+G(x_m^{(1)},F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)})+G(x_m^{(2)},F(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)}),$$
$$F(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)})+\ldots+G(x_m^{(n)},F(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}),F(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}))\}.$$

Let for all m, n ≥ 0,

$$\delta_{m+1} = G(x_m^{(1)},x_{m+1}^{(1)},x_{m+1}^{(1)})+G(x_m^{(2)},x_{m+1}^{(2)},x_{m+1}^{(2)})+\ldots+G(x_m^{(n)},x_{m+1}^{(n)},x_{m+1}^{(n)}). \tag{25}$$

we have

$$\delta_{m+1} \leq \tfrac{n}{2n}\{\delta_{m+1}+\delta_m\} - n\phi\{\tfrac{\delta_{m+1}+\delta_m}{2n}\}$$

$$2\delta_{m+1} \leq \delta_{m+1}+\delta_m - 2n\phi\{\tfrac{\delta_{m+1}+\delta_m}{2n}\}$$

$$\delta_{m+1} \leq \delta_m - 2n\phi\{\frac{\delta_{m+1}+\delta_m}{2n}\} \tag{26}$$

$$\Rightarrow \delta_{m+1} \leq \delta_m. \text{ (by the property of } \phi\text{- function)}$$

Therefore the sequence $\{\delta_m\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists $\delta \geq 0$ such that $lim_{m\to\infty}\delta_m = \delta$. Assume $\delta > 0$. Then by again (3.23), taking limit $m \to \infty$, we get

$$lim_{m\to\infty}\delta_{m+1} \leq lim_{m\to\infty}\delta_m - 2n\phi\{lim_{m\to\infty}\tfrac{\delta_{m+1}+\delta_m}{2n}\}$$

$$\delta \leq \delta - 2n\phi\{\tfrac{\delta+\delta}{2n}\},$$

which is a contradiction. Hence $lim_{m\to\infty}\delta_m = 0$

$$\Rightarrow lim_{m\to\infty}\{G(x_m^{(1)},x_{m+1}^{(1)},x_{m+1}^{(1)})+G(x_m^{(2)},x_{m+1}^{(2)},x_{m+1}^{(2)})+\ldots+G(x_m^{(n)},x_{m+1}^{(n)},x_{m+1}^{(n)})\} = 0.$$

Hence

$$\left.\begin{aligned} lim_{m\to\infty}\{G(x_m^{(1)},x_{m+1}^{(1)},x_{m+1}^{(1)})\} &= 0\\ lim_{m\to\infty}\{G(x_m^{(2)},x_{m+1}^{(2)},x_{m+1}^{(2)})\} &= 0\\ &\vdots\\ lim_{m\to\infty}\{G(x_m^{(n)},x_{m+1}^{(n)},x_{m+1}^{(n)})\} &= 0.\end{aligned}\right\} \tag{27}$$

Next we have to show that $\{x_m^{(1)}\}$, $\{x_m^{(2)}\}$, $\{x_m^{(3)}\}$, $\ldots$ and $\{x_m^{(n)}\}$ are Cauchy sequence. If possible, let at least one of $\{x_m^{(1)}\}$,... and $\{x_m^{(n)}\}$ be not a Cauchy sequence. Then there exist $\varepsilon > 0$ and sequence of natural numbers $\{n(k)\}$ and

$\{l(k)\}$ for which $n(k) > l(k) \geq k$, and such that for all $k \geq 1$, either

$$\left.\begin{array}{r} G(x^{(1)}_{l(k)},x^{(1)}_{n(k)},x^{(1)}_{n(k)})\} \geq \varepsilon \ \ or \\ G(x^{(2)}_{l(k)},x^{(2)}_{n(k)},x^{(2)}_{n(k)})\} \geq \varepsilon \ \ or \\ G(x^{(n)}_{l(k)},x^{(n)}_{n(k)},x^{(n)}_{n(k)})\} \geq \varepsilon \ \ or \\ \vdots \qquad\qquad \\ G(x^{(n)}_{l(k)},x^{(n)}_{n(k)},x^{(n)}_{n(k)})\} \geq \varepsilon. \end{array}\right\} \tag{28}$$

Then for all $k \geq 1$,

$$g_k = G(x^{(1)}_{l(k)},x^{(1)}_{n(k)},x^{(1)}_{n(k)}) + G(x^{(2)}_{l(k)},x^{(2)}_{n(k)},x^{(2)}_{n(k)}) + \ldots + G(x^{(n)}_{l(k)},x^{(n)}_{n(k)},x^{(n)}_{n(k)}) \geq \varepsilon. \tag{29}$$

Now corresponding to $l(k)$ we can choose $n(k)$ to be the smallest positive integer for which (3.28) holds. Then for all $k \geq 1$,

$$G(x^{(1)}_{l(k)},x^{(1)}_{n(k)-1},x^{(1)}_{n(k-1)}) + G(x^{(2)}_{l(k)},x^{(2)}_{n(k)-1},x^{(2)}_{n(k)-1}) + \ldots + G(x^{(n)}_{l(k)},x^{(n)}_{n(k)-1},x^{(n)}_{n(k)-1}) < \varepsilon. \tag{30}$$

Again, from (3.28), using triangle inequality, for all $k \geq 1$, we have

$$\begin{aligned} \varepsilon \leq g_k &= G(x^{(1)}_{l(k)},x^{(1)}_{n(k)},x^{(1)}_{n(k)}) + G(x^{(2)}_{l(k)},x^{(2)}_{n(k)},x^{(2)}_{n(k)}) + \ldots + G(x^{(n)}_{l(k)},x^{(n)}_{n(k)},x^{(n)}_{n(k)}) \\ &\leq G(x^{(1)}_{l(k)},x^{(1)}_{n(k-1)},x^{(1)}_{n(k)-1}) + G(x^{(1)}_{n(k)-1},x^{(1)}_{n(k)},x^{(1)}_{n(k)}) + G(x^{(2)}_{l(k)},x^{(2)}_{n(k-1)},x^{(2)}_{n(k)-1}) \\ &+ G(x^{(2)}_{n(k)-1},x^{(2)}_{n(k)},x^{(2)}_{n(k)}) + \ldots + G(x^{(n)}_{l(k)},x^{(n)}_{n(k)-1},x^{(n)}_{n(k)-1}) + G(x^{(n)-1}_{l(k)},x^{(n)}_{n(k)},x^{(n)}_{n(k)}) \\ &\leq G(x^{(1)}_{l(k)},x^{(1)}_{n(k-1)},x^{(1)}_{n(k)-1}) + G(x^{(2)}_{l(k)},x^{(2)}_{n(k-1)},x^{(2)}_{n(k)-1}) + \ldots + G(x^{(n)}_{l(k)},x^{(n)}_{n(k)-1},x^{(n)}_{n(k)-1}) \\ &+ A^1_{n(k)-1} + A^2_{n(k)-1} + \ldots + A^n_{n(k)-1} < \varepsilon + A^1_{n(k)-1} + A^2_{n(k)-1} + \ldots + A^n_{n(k)-1}. \end{aligned}$$

Taking limit $k \to \infty$, we have

$$lim_{k\to\infty} g_k = \varepsilon. \tag{31}$$

Again, for all $k \geq 1$ and by using the triangle inequality, we have

$$\begin{aligned} g_k &= G(x^{(1)}_{l(k)},x^{(1)}_{n(k)},x^{(1)}_{n(k)}) + G(x^{(2)}_{l(k)},x^{(2)}_{n(k)},x^{(2)}_{n(k)}) + \ldots + G(x^{(n)}_{l(k)},x^{(n)}_{n(k)},x^{(n)}_{n(k)}) \\ &\leq G(x^{(1)}_{l(k)},x^{(1)}_{l(k)+1},x^{(1)}_{l(k)+1}) + G(x^{(1)}_{l(k)+1},x^{(1)}_{n(k)},x^{(1)}_{n(k)}) + G(x^{(2)}_{l(k)},x^{(2)}_{l(k)+1},x^{(2)}_{l(k)+1}) \\ &+ G(x^{(2)}_{l(k)+1},x^{(2)}_{n(k)},x^{(2)}_{n(k)}) + \ldots + G(x^{(n)}_{l(k)},x^{(n)}_{l(k)+1},x^{(n)}_{l(k)+1}) + G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k)},x^{(n)}_{n(k)}) \\ &\leq G(x^{(1)}_{l(k)},x^{(1)}_{l(k)+1},x^{(1)}_{l(k)+1}) + G(x^{(1)}_{l(k)+1},x^{(1)}_{n(k)+1},x^{(1)}_{n(k)+1}) + G(x^{(1)}_{n(k)+1},x^{(1)}_{n(k)},x^{(1)}_{n(k)}) \\ &+ \ldots + G(x^{(n)}_{l(k)},x^{(n)}_{l(k)+1},x^{(n)}_{l(k)+1}) + G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k)+1},x^{(n)}_{n(k)+1}) + G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k)},x^{(n)}_{n(k)}). \end{aligned}$$

Taking limit $k \to \infty$, we have

$$\varepsilon = lim_{k\to\infty} g_k \leq lim_{k\to\infty}(G(x^{(1)}_{l(k)+1},x^{(1)}_{n(k+1)},x^{(1)}_{n(k)+1}) + \nu + G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k+1)},x^{(n)}_{n(k)+1})). \tag{32}$$

Now again,

$$
\begin{aligned}
&G(x^{(1)}_{l(k)+1},x^{(1)}_{n(k)+1},x^{(1)}_{n(k)+1}+G(x^{(2)}_{l(k)+1},x^{(2)}_{n(k)+1},x^{(2)}_{n(k)+1})+\ldots+G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k)+1},x^{(n)}_{n(k)+1})\\
&\leq G(x^{(1)}_{l(k)+1},x^{(1)}_{l(k)},x^{(1)}_{l(k)})+G(x^{(1)}_{l(k)},x^{(1)}_{n(k)},x^{(1)}_{n(k)})+G(x^{(1)}_{n(k)},x^{(1)}_{n(k)+1},x^{(1)}_{n(k)+1})+\ldots\\
&+G(x^{(n)}_{l(k)+1},x^{(n)}_{l(k)},x^{(n)}_{l(k)})+G(x^{(n)}_{l(k)},x^{(n)}_{n(k)},x^{(n)}_{n(k)})+G(x^{(n)}_{n(k)},x^{(n)}_{n(k)+1},x^{(n)}_{n(k)+1}).
\end{aligned}
$$

Taking limit $k\to\infty$, we have

$$
lim_{k\to\infty}(G(x^{(1)}_{l(k)+1},x^{(1)}_{n(k+1)},x^{(1)}_{n(k)+1})+\ldots+G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k+1)},x^{(n)}_{n(k)+1}))\leq lim_{k\to\infty}g_k=\varepsilon. \tag{33}
$$

By above two equations, we get

$$
lim_{k\to\infty}(G(x^{(1)}_{l(k)+1},x^{(1)}_{n(k+1)},x^{(1)}_{n(k)+1})+\ldots+G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k+1)},x^{(n)}_{n(k)+1}))=\varepsilon. \tag{34}
$$

For all $k\geq 1$, we obtain,

$$
\begin{aligned}
&G(x^{(1)}_{l(k)+1},x^{(1)}_{n(k+1)},x^{(1)}_{n(k)+1})=G(F(x^{(1)}_{l(k)},x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)}),F(x^{(1)}_{n(k)},x^{(2)}_{n(k)}),\ldots,x^{(n)}_{n(k)}),\\
&F(x^{(1)}_{n(k)},x^{(2)}_{n(k)}),\ldots,x^{(n)}_{n(k)}))\leq M(x^{(1)}_{l(k)},x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{n(k)},x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)})-\\
&\phi\{M(x^{(1)}_{l(k)},x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{n(k)},x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)})\}\\
&G(x^{(2)}_{l(k)+1},x^{(2)}_{n(k+1)},x^{(2)}_{n(k)+1})=G(F(x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)}),x^{(1)}_{l(k)}),F(x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)}),x^{(1)}_{n(k)})\\
&F(x^{(2)}_{n(k)}),\ldots,x^{(n)}_{n(k)},F(x^{(1)}_{n(k)}))\leq M(x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{l(k)},x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)},x^{(1)}_{n(k)})-\\
&\phi\{M(x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{l(k)},x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)},x^{(1)}_{n(k)})\}\\
&\vdots\\
&G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k+1)},x^{(n)}_{n(k)+1})=G(F(x^{(n)}_{l(k)},x^{(1)}_{l(k)},\ldots,x^{(n-1)}_{l(k)}),F(x^{(n)}_{n(k)},x^{(1)}_{n(k)}),\ldots,x^{(n-1)}_{n(k)}),\\
&F(x^{(n)}_{n(k)},x^{(1)}_{n(k)}),\ldots,x^{(n-1)}_{n(k)}))\leq M(x^{(n)}_{l(k)},x^{(1)}_{l(k)},\ldots,x^{(n-1)}_{l(k)},x^{(n)}_{n(k)},x^{(1)}_{n(k)},\ldots,x^{(n-1)}_{n(k)})-\\
&\phi\{M(x^{(n)}_{l(k)},x^{(1)}_{l(k)},\ldots,x^{(n-1)}_{l(k)},x^{(n)}_{n(k)},x^{(1)}_{n(k)},\ldots,x^{(n-1)}_{n(k)})\},
\end{aligned}
$$

where

$$
\begin{aligned}
&M(x^{(1)}_{l(k)},x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{n(k)},x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)})=\frac{1}{2n}\{G(x^{(1)}_{l(k)},F(x^{(1)}_{l(k)},x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)}),\\
&F(x^{(1)}_{l(k)},x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)}))+G(x^{(2)}_{l(k)},F(x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{l(k)}),F(x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{l(k)}))+\ldots+\\
&G(x^{(n)}_{l(k)},F(x^{(n)}_{l(k)},x^{(1)}_{l(k)},\ldots,x^{(n-1)}_{l(k)}),F(x^{(n)}_{l(k)},x^{(1)}_{l(k)},\ldots,x^{(n-1)}_{l(k)}))+G(x^{(1)}_{n(k)},F(x^{(1)}_{n(k)},x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)}),\\
&F(x^{(1)}_{n(k)},x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)}))+G(x^{(2)}_{n(k)},F(x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)},x^{(1)}_{n(k)}),F(x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)},x^{(1)}_{n(k)}))+\ldots+\\
&G(x^{(n)}_{n(k)},F(x^{(n)}_{n(k)},x^{(1)}_{n(k)},\ldots,x^{(n-1)}_{n(k)}),F(x^{(n)}_{n(k)},x^{(1)}_{n(k)},\ldots,x^{(n-1)}_{n(k)}))\}\\
&=\frac{1}{2n}\{G(x^{(1)}_{l(k)},x^{(1)}_{l(k)+1},x^{(1)}_{l(k)+1})+G(x^{(2)}_{l(k)},x^{(2)}_{l(k)+1},x^{(2)}_{l(k)+1})+\ldots+G(x^{(n)}_{l(k)},x^{(n)}_{l(k)+1},x^{(n)}_{l(k)+1})\\
&+G(x^{(1)}_{n(k)},x^{(1)}_{n(k)+1},x^{(1)}_{n(k)+1})+G(x^{(2)}_{n(k)},x^{(2)}_{n(k)+1},x^{(2)}_{n(k)+1})+\ldots+G(x^{(n)}_{n(k)},x^{(n)}_{n(k)+1},x^{(n)}_{n(k)+1})\}.
\end{aligned}
$$

Adding above equations and taking limit $k \to \infty$, we have

$$lim_{\to\infty}\{G(x^{(1)}_{l(k)+1},x^{(1)}_{n(k+1)},x^{(1)}_{n(k)+1})+G(x^{(2)}_{l(k)+1},x^{(2)}_{n(k+1)},x^{(2)}_{n(k)+1})+\ldots+G(x^{(n)}_{l(k)+1},x^{(n)}_{n(k+1)},x^{(n)}_{n(k)+1})\}$$

$$\leq n\{lim_{k\to\infty}M(x^{(1)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{n(k)},\ldots,x^{(n)}_{n(k)}\}-n\{\phi(M(x^{(1)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{n(k)},\ldots,x^{(n)}_{n(k)}))\}$$

where

$$lim_{k\to\infty}M(x^{(1)}_{l(k)},x^{(2)}_{l(k)},\ldots,x^{(n)}_{l(k)},x^{(1)}_{n(k)},x^{(2)}_{n(k)},\ldots,x^{(n)}_{n(k)}\}=0. \tag{35}$$

we obtain $\varepsilon \leq 0$, which is contradiction. Therefore $\{x^{(1)}_{m)}\}$, $\{x^{(2)}_{m)}\}$,... an $\{x^{(n)}_{m)}\}$ are cauchy sequence in X and hence they are convergent in the complete G-metric space. Let

$$\left.\begin{array}{ll} x^{(1)}_m \to x^1 as & m\to\infty \\ x^{(2)}_m \to x^2 as & m\to\infty \\ x^{(3)}_m \to x^3 as & m\to\infty \\ \vdots & \\ x^{(n)}_m \to x^n as & m\to\infty. \end{array}\right\} \tag{36}$$

Step 4: Now we have to show that F has n-tupled fixed points.
Let the condition (2) of the theorem holds. that is F is continuous. From (3.4) and (3.35),we have we have

$$x^{(1)}=lim_{m\to\infty}x^{(1)}_{m+1}=F(lim_{m\to\infty}x^{(1)}_m,lim_{m\to\infty}x^{(2)}_m,\ldots,lim_{m\to\infty}x^{(n)}_m)=F(x^{(1)},x^{(2)},\ldots,x^{(n)})$$

$$x^{(2)}=lim_{m\to\infty}x^{(2)}_{m+1}=F(lim_{m\to\infty}x^{(2)}_m,\ldots,lim_{m\to\infty}x^{(n)}_m,lim_{m\to\infty}x^{(1)}_m)=F(x^{(2)},\ldots,x^{(n)},x^{(1)})$$
$$\vdots$$
$$x^{(n)}=lim_{m\to\infty}x^{(n)}_{m+1}=F(lim_{m\to\infty}x^{(n)}_m,lim_{m\to\infty}x^{(1)}_m,\ldots,lim_{m\to\infty}x^{(n-1)}_m)=F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}).$$

Next we assume that the condition (3) holds. Since $\{x^{(i)}_m\}$ is non-decreasing or non-increasing according as i is odd or even and $x^{(n)}_m \to x^{(n)}$ (as $m\to\infty$). Then by assumption (b) we have for all m,

$$x^{(i)}_m \preceq x^{(i)} \text{ when i is odd.}$$
$$x^{(i)}_m \succeq x^{(i)} \text{ when i is even.}$$

Consider now,

$$\begin{aligned}&G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),x^{(1)}_{m+1},x^{(1)}_{m+1})=G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),F(x^{(1)}_m,x^{(2)}_m),\ldots,x^{(n)}_m),\\ &F(x^{(1)}_m,x^{(2)}_m),\ldots,x^{(n)}_m))\leq M(x^{(1)},x^{(2)},\ldots,x^{(n)},x^{(1)}_m,x^{(2)}_m,\ldots,x^{(n)}_m)\\ &-\phi\{M(x^{(1)},x^{(2)},\ldots+x^{(n)},x^{(1)}_m,x^{(2)}_m,\ldots,x^{(n)}_m)\}\\ &G(F(x^{(2)},\ldots,x^{(n)},x^{(1)}),x^{(2)}_{m+1},x^{(2)}_{m+1})=G(F(x^{(2)},\ldots,x^{(n)},x^{(1)}),F(x^{(2)}_m),\ldots,x^{(n)}_m,x^{(1)}_m),\\ &F(x^{(2)}_m),\ldots,x^{(n)}_m,x^{(1)}_m))\leq M(x^{(2)},\ldots x^{(n)},x^{(1)},x^{(2)}_m,\ldots,x^{(n)}_m,x^{(1)}_m)\end{aligned}$$

$$-\phi\{M(x^{(2)},\ldots,x^{(n)},x^{(1)},x_m^{(2)},...,x_m^{(n)},x_m^{(1)}))\}$$
$$\vdots$$
$$G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),x_{m+1}^{(n)},x_{m+1}^{(n)}) = G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),F(x_m^{(n)},x_m^{(1)}),\ldots,x_m^{(n-1)}),$$
$$F(x_m^{(n)},x_m^{(1)}),\ldots,x_m^{(n-1)})) \leq M(x^{(n)},x^{(1)},\ldots,x^{(n-1)},x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)})$$
$$-\phi\{M(x^{(n)},x^{(1)},\ldots,x^{(n-1)},x_m^{(n)},x_m^{(1)},...,x_m^{(n-1)})\},$$

where

$$M(x^{(1)},x^{(2)},\ldots,x^{(n)},x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}) = \frac{1}{2n}\{G(x^{(1)},F(x^{(1)},x^{(2)},\ldots,x^{(n)}),$$
$$F(x^{(1)},x^{(2)},\ldots,x^{(n)})) + \ldots + G(x^{(n)},F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),F(x^{(n)},x^{(1)},\ldots,x^{(n-1)})$$
$$+G(x_m^{(1)},F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),F(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}) + \ldots + G(x_m^{(n)},F(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}),$$
$$F(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)})\}.$$

Now adding above, we have

$$G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),x_{m+1}^{(1)},x_{m+1}^{(1)}) + G(F(x^{(2)},\ldots,x^{(n)},x^{(1)}),x_{m+1}^{(2)},x_{m+1}^{(2)}) + \ldots$$
$$+G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),x_{m+1}^{(n)},x_{m+1}^{(n)}) \leq nM(x^{(n)},x^{(1)},\ldots,x^{(n-1)},x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)})$$
$$-n\phi\{M(x^{(n)},x^{(1)},\ldots,x^{(n-1)},x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)})\} < nM(x^{(n)},x^{(1)},\ldots,x^{(n-1)},x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}).$$

Taking limit $m \to \infty$, we have

$$lim_{m\to\infty}G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),x_{m+1}^{(1)},x_{m+1}^{(1)}) + G(F(x^{(2)},\ldots,x^{(n)},x^{(1)}),x_{m+1}^{(2)},x_{m+1}^{(2)}) + \ldots$$
$$+G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),x_{m+1}^{(n)},x_{m+1}^{(n)}) < \frac{n}{2n}lim_{m\to\infty}\{G(x^{(1)},F(x^{(1)},x^{(2)},\ldots,x^{(n)}),$$
$$F(x^{(1)},x^{(2)},\ldots,x^{(n)})) + \ldots + G(x^{(n)},F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),F(x^{(n)},x^{(1)},...,x^{(n-1)}\}.$$
$$\Rightarrow G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),x^{(1)},x^{(1)}) + G(F(x^{(2)},\ldots,x^{(n)},x^{(1)}),x^{(2)},x^{(2)}) + \ldots$$
$$+G(F(x^{(n)},x^{(1)},\ldots,x^{(n-1)}),x^{(n)},x^{(n)}) = 0.$$

Therefore,

$$F(x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)}) = x^{(1)}$$
$$F(x^{(2)},x^{(3)},\ldots,x^{(n)},x^{(1)}) = x^{(2)}$$
$$F(x^{(3)},\ldots,x^{(n)},x^{(1)},x^{(2)}) = x^{(3)}$$
$$\vdots$$
$$F(x^{(n)},x^{(1)},\ldots,x^{(n-2)},x^{(n-1)}) = x^{(n)}.$$

Thus F has n-tupled fixed point in X.

Corollary 3.3. Let $(X,G,\preceq)$ be partially ordered complete G-metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has mixed g-monotone property on X and satisfies the following conditions:

$$G(F(x^{(1)},x^{(2)},...,x^{(n)}),F(y^{(1)},y^{(2)},...,y^{(n)}),F(z^{(1)},z^{(2)},...,z^{(n)})) \leq$$
$$k\{\frac{G(gx^{(1)},gy^{(1)},gz^{(1)})+G(gx^{(2)},gy^{(2)},gz^{(2)})+...+G(gx^{(n)},gy^{(n)},gz^{(n)})}{n}]\} \quad (37)$$

for all $x^{(1)},x^{(2)},x^{(3)},...,x^{(n)},y^{(1)},y^{(2)},y^{(3)},...,y^{(n)},z^{(1)},z^{(2)},z^{(3)},...,z^{(n)} \in X$ with $gz^{(1)} \preceq gy^{(1)} \preceq gx^{(1)}$, $gz^{(2)} \succeq gy^{(2)} \succeq gx^{(2)}...$ $gz^{(n)} \succeq gy^{(n)} \succeq gx^{(n)}$, where $gy^{(1)} \neq gz^{(1)}$, $gy^{(2)} \neq gz^{(2)},...,gy^{(n)} \neq gz^{(n)}$,where $k \in [0,1)$. We assume the following hypothesis:

1. $F(X^n) \subseteq g(X)$, $g(X)$ is complete, g is continuous and F is g-compatible.
2. F is continuous or
3. (a) if a non-decreasing sequence $\{x_m\} \to x$, then $gx_m \preceq gx$, for all $m \geq 0$.
   (b) if a non-increasing sequence $\{y_m\} \to y$, then $gy \preceq gy_m$, for all $m \geq 0$.

and if there are $x_0^{(1)},x_0^{(2)},x_0^{(3)},...,x_0^{(n)} \in X$ such that $gx_0^{(1)} \preceq F(x_0^{(1)},x_0^{(2)},x_0^{(3)},...,x_0^{(n)})$, $gx_0^{(2)} \succeq F(x_0^{(2)},x_0^{(3)},...,x_0^{(n)},x_0^{(1)})$, $gx_0^{(3)} \preceq F(x_0^{(3)},...,x_0^{(n)},x_0^{(1)},x_0^{(2)}),\ldots,gx_0^{(n)} \succeq F(x_0^{(n)},x_0^{(1)},\ldots,x_0^{(n-2)},x_0^{(n-1)})$. Then F and g have n-tupled coincidence point in X.

Proof. Follow from Theorem 3.1 by putting $\phi(t) = (1-k)t$, where $k \in (0,1)$.

Corollary 3.4. Let $(X,G,\preceq)$ be partially ordered complete G-metric space. Let $F : X^n \to X$ be a mapping such that F has mixed g-monotone property on X and satisfies the following conditions:

$$\begin{aligned}&G(F(x^{(1)},x^{(2)},x^{(3)},...,x^{(n)}),F(y^{(1)},y^{(2)},y^{(3)},...,y^{(n)}),F(z^{(1)},z^{(2)},z^{(3)},...,z^{(n)}))\\&\leq\{\frac{G(x^{(1)},y^{(1)},z^{(1)})+...+G(x^{(n)},y^{(n)},z^{(n)})}{n}\}-\phi\{\frac{G(x^{(1)},y^{(1)},z^{(1)})+...+G(x^{(n)},y^{(n)},z^{(n)})}{n}\},\end{aligned} \quad (38)$$

for all $x^{(1)},x^{(2)},x^{(3)},...,x^{(n)},y^{(1)},y^{(2)},y^{(3)},...,y^{(n)},z^{(1)},z^{(2)},z^{(3)},...,z^{(n)} \in X$ with $z^{(1)} \preceq y^{(1)} \preceq x^{(1)}$, $z^{(2)} \succeq y^{(2)} \succeq x^{(2)}...$ $z^{(n)} \succeq y^{(n)} \succeq x^{(n)}$, where $y^{(1)} \neq z^{(1)}$, $y^{(2)} \neq z^{(2)},...,y^{(n)} \neq z^{(n)}$ and $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > 0$ for all $t \in (0,\infty)$. We assume the following hypothesis:

1. $F(X^n) \subseteq (X)$,
2. F is continuous or
3. (a) if a non-decreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$, for all $m \geq 0$.
   (b) if a non-increasing sequence $\{y_m\} \to y$, then $y \preceq y_m$, for all $m \geq 0$.

and if there are $x_0^{(1)},x_0^{(2)},x_0^{(3)},...,x_0^{(n)} \in X$ such that $x_0^{(1)} \preceq F(x_0^{(1)},x_0^{(2)},x_0^{(3)},...,x_0^{(n)})$, $x_0^{(2)} \succeq F(x_0^{(2)},x_0^{(3)},...,x_0^{(n)},x_0^{(1)})$, $x_0^{(3)} \preceq F(x_0^{(3)},...,x_0^{(n)},x_0^{(1)},x_0^{(2)}),\ldots,x_0^{(n)} \succeq F(x_0^{(n)},x_0^{(1)},\ldots,x_0^{(n-2)},x_0^{(n-1)})$. Then F has n-tupled fixed point in X.

Proof. Follow from Theorem 3.1 by taking g is an identity map.

Corollary 3.5. Let $(X,G,\preceq)$ be partially ordered complete G-metric space. Let $F : X^n \to X$ be a mapping such that F has mixed g-monotone property on X and satisfies the following conditions:

$$\begin{aligned}&G(F(x^{(1)},x^{(2)},x^{(3)},...,x^{(n)}),F(y^{(1)},y^{(2)},y^{(3)},...,y^{(n)}),F(z^{(1)},z^{(2)},z^{(3)},...,z^{(n)})) \leq\\&k\{\tfrac{G(x^{(1)},y^{(1)},z^{(1)})+G(x^{(2)},y^{(2)},z^{(2)})+...+G(x^{(n)},y^{(n)},z^{(n)})}{n}\}\end{aligned} \quad (39)$$

for all $x^{(1)},x^{(2)},x^{(3)},...,x^{(n)},y^{(1)},y^{(2)},y^{(3)},...,y^{(n)},z^{(1)},z^{(2)},z^{(3)},...,z^{(n)} \in X$ with $z^{(1)} \preceq y^{(1)} \preceq x^{(1)}$, $z^{(2)} \succeq y^{(2)} \succeq x^{(2)}...$ $z^{(n)} \succeq y^{(n)} \succeq x^{(n)}$, where $y^{(1)} \neq z^{(1)}$, $y^{(2)} \neq z^{(2)},...,y^{(n)} \neq z^{(n)}$, where $k \in [0,1)$. We assume the following hypothesis:

1. $F(X^n) \subseteq (X)$,
2. F is continuous or
3. (a) if a non-decreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$, for all $m \geq 0$.
   (b) if a non-increasing sequence $\{y_m\} \to y$, then $y \preceq y_m$, for all $m \geq 0$.

and if there are $x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)} \in X$ such that $x_0^{(1)} \preceq F(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$, $x_0^{(2)} \succeq F(x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}, x_0^{(1)})$, $x_0^{(3)} \preceq F(x_0^{(3)}, ..., x_0^{(n)}, x_0^{(1)}, x_0^{(2)}), \ldots, x_0^{(n)} \succeq F(x_0^{(n)}, x_0^{(1)}, \ldots, x_0^{(n-2)}, x_0^{(n-1)})$. Then F has n-tupled fixed point in X.

Proof. Follow from Corollary 3.4 by putting $\phi(t) = (1-k)t$, where $k \in (0,1)$.

Corollary 3.6. Let $(X, G, \preceq)$ be partially ordered complete G-metric space. Let $F : X \times X \times X \ldots \times X \to X$ be a mapping such that F has mixed monotone property and satisfies the following conditions:

$$G(F(x^{(1)}, x^{(2)}, \ldots, x^{(n)}), F(y^{(1)}, y^{(2)}, \ldots, y^{(n)}), F(y^{(1)}, y^{(2)}, \ldots, y^{(n)})) \leq k\{M(x^{(1)}, x^{(2)}, \ldots, x^{(n)}, y^{(1)}, y^{(2)}, \ldots, y^{(n)})\} \quad (40)$$

where

$$\begin{aligned} M(x^{(1)}, x^{(2)}, \ldots, x^{(n)}, y^{(1)}, y^{(2)}, \ldots, y^{(n)}) &= \frac{1}{2n}\{G(x^{(1)}, F(x^{(1)}, x^{(2)}, \ldots, x^{(n)}), F(x^{(1)}, x^{(2)}, \ldots, x^{(n)})) \\ &+ G(x^{(2)}, F(x^{(2)}, \ldots, x^{(n)}, x^{(1)}), F(x^{(2)}, \ldots, x^{(n)}, x^{(1)})) + \ldots + G(x^{(n)}, F(x^{(n)}, x^{(1)}, \ldots, x^{(n-1)}), F(x^{(n)}, x^{(1)}, \ldots, x^{(n-1)})) \\ &+ G(y^{(1)}, F(y^{(1)}, y^{(2)}, \ldots, y^{(n)}), F(y^{(1)}, y^{(2)}, \ldots, y^{(n)})) + G(y^{(2)}, F(y^{(2)}, \ldots, y^{(n)}, y^{(1)}), F(y^{(2)}, \ldots, y^{(n)}, y^{(1)})) \\ &+ \ldots + G(y^{(n)}, F(y^{(n)}, y^{(1)}, \ldots, y^{(n-1)}), F(y^{(n)}, y^{(1)}, \ldots, y^{(n-1)}))\} \end{aligned}$$

for all $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in X$ and $y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(n)} \in X$ with $y^{(1)} \preceq x^{(1)}$, $x^{(2)} \preceq y^{(2)} \ldots x^{(n)} \preceq y^{(n)}$. Also suppose that

1. F is continuous or
2. (a) if a non-decreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$, for all m.
   (b) if a non-increasing sequence $\{y_m\} \to y$, then $y \preceq y_m$, for all m.

and if there are $x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)} \in X$ such that $x_0^{(1)} \preceq F(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$, $x_0^{(2)} \succeq F(x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}, x_0^{(1)})$, $x_0^{(3)} \preceq F(x_0^{(3)}, ..., x_0^{(n)}, x_0^{(1)}, x_0^{(2)}), \ldots, x_0^{(n)} \succeq F(x_0^{(n)}, x_0^{(1)}, \ldots, x_0^{(n-2)}, x_0^{(n-1)})$ (when n is even). Then F has n-tupled fixed point in X.

Proof. Follow from Theorem 3.2 by putting $\phi(t) = (1-k)t$, where $k \in (0,1)$.

4 Applications

Now we produce one example to support our Theorem 3.1.

Let $X = R$ be ordered by the following relation

$$x \preceq y \Leftrightarrow x = y \text{ or } (x, y \in [0,1] \text{ and } x \leq y.)$$

Let G -metric on X be defined by

$$G(x,y,z) = |x-y| + |y-z| + |z-x|$$

Then $(X, G, \preceq)$ is a complete regular ordered G- metric space. Let $F : X^n \to X$ be defined by

$$F(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}) = \begin{cases} \frac{(x^{(1)})^2 - (x^{(2)})^2 + (x^{(3)})^3 -, \ldots, -(x^{(n)})^2}{2n} & \text{, if } x^{i+1} \preceq x^i \text{ where } i = 1, 2, .., n-1 \\ 0 & \text{, otherwise} \end{cases}$$

Let $g(x)=x^2$, where $x\in X$, F is g compatible we can check (Definition 2.18) by considering the sequences $x_m^{(1)}=\{\frac{1}{m}:m\in N\}$, $x_m^{(2)}=\{\frac{-1}{m}:m\in N\}$,...,$x_m^{(n)}=\{\frac{-1}{m}:m\in N\}$. Let $\phi(t)=\frac{t}{2}$ for $t\in[0,\infty)$.

Now choose $(x_0^{(1)},x_0^{(2)},x_0^{(3)},...,x_0^{(n)})=(0,c,0,c,c,...,c)$ $(c>0)$ Then

$$\left.\begin{aligned} gx_0^{(1)}&=0=F(x_0^{(1)},x_0^{(2)},x_0^{(3)},\dots,x_0^{(n)})=gx_1^{(1)}\\ gx_0^{(2)}&=c^2\succeq F(x_0^{(2)},x_0^{(3)},\dots,x_0^{(n)},x_0^{(1)})=0=gx_1^{(2)}\\ gx_0^{(3)}&=0=F(x_0^{(3)},\dots,x_0^{(n)},x_0^{(1)},x_0^{(2)})=0=gx_1^{(3)}\\ gx_0^{(n)}&=c^2\succeq F(x_0^{(n)},x_0^{(1)},\dots,x_0^{(n-2)},x_0^{(n-1)})=0=gx_1^{(n)}\end{aligned}\right\}.$$

We next verify inequality (3.1) of Theorem 3.1. we take $x^{(1)},x^{(2)},\dots,x^{(n)},y^{(1)},y^{(2)},\dots,y^{(n)},z^{(1)},z^{(2)}$ $,\dots,z^{(n)}\in X$ such that

$$z^{(1)}\preceq y^{(1)}\preceq x^{(1)},\ z^{(2)}\succeq y^{(2)}\succeq x^{(2)}...\ z^{(n)}\succeq y^{(n)}\succeq x^{(n)}.$$

The following eight cases arises:

Case 1. Let $x^{(1)},x^{(2)},x^{(3)},\dots,x^{(n)},y^{(1)},y^{(2)},y^{(3)},\dots,y^{(n)},z^{(1)},z^{(2)},z^{(3)},\dots,z^{(n)}\in X$ such that $x^{(i+1)}\preceq x^{(i)}$, $y^{(i+1)}\preceq y^{(i)}$, $z^{(i+1)}\preceq z^{(i)}$ for $i=1,3,...,n-1$. Then

$$\begin{aligned}
&G(F(x^{(1)},x^{(2)},\dots,x^{(n)}),F(y^{(1)},y^{(2)},\dots,y^{(n)}),F(z^{(1)},z^{(2)},\dots,z^{(n)}))=|F(x^{(1)},x^{(2)},\dots,x^{(n)})-F(y^{(1)},y^{(2)}\\
&,\dots,y^{(n)})|+|F(y^{(1)},y^{(2)},\dots,y^{(n)})-F(z^{(1)},z^{(2)},\dots,z^{(n)})|+|F(z^{(1)},z^{(2)},\dots,z^{(n)})-F(x^{(1)},x^{(2)},\dots,x^{(n)})|\\
&=|\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\dots,-(x^{(n)})^2}{2n}-\frac{(y^{(1)})^2-(y^{(2)})^2+(y^{(3)})^3-,\dots,-(y^{(n)})^2}{2n}|\\
&+|\frac{(y^{(1)})^2-(y^{(2)})^2+(y^{(3)})^3-,\dots,-(y^{(n)})^2}{2n}\\
&-\frac{(z^{(1)})^2-(z^{(2)})^2+(z^{(3)})^3-,\dots,-(z^{(n)})^2}{2n}|+|\frac{(z^{(1)})^2-(z^{(2)})^2+(z^{(3)})^3-,\dots,-(z^{(n)})^2}{2n}\\
&-\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\dots,-(x^{(n)})^2}{2n}|\\
&\le\frac{1}{2n}\{|(x^{(1)})^2-(y^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+...+|(x^{(n)})^2-(y^{(n)})^2|+|(y^{(1)})^2-(z^{(1)})^2|+|(y^{(2)})^2-(z^{(2)})^2|+...\\
&+|(y^{(n)})^2-(z^{(n)})^2|+|(z^{(1)})^2-(x^{(1)})^2|+|(z^{(2)})^2-(x^{(2)})^2|+...+|(z^{(n)})^2-(x^{(n)})^2|\}\\
&=\frac{1}{n}\{|(x^{(1)})^2-(y^{(1)})^2|+|(y^{(1)})^2-(z^{(1)})^2|+(z^{(1)})^2-(x^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+|(y^{(2)})^2-(z^{(2)})^2|+|(z^{(2)})^2\\
&-(x^{(2)})^2|+...+|(x^{(n)})^2-(y^{(n)})^2|+|(y^{(n)})^2-(z^{(n)})^2|+|(z^{(n)})^2-(x^{(n)})^2|\}-\frac{1}{2n}\{|(x^{(1)})^2-(y^{(1)})^2|+|(y^{(1)})^2-(z^{(1)})^2|\\
&+(z^{(1)})^2-(x^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+|(y^{(2)})^2-(z^{(2)})^2|+|(z^{(2)})^2-(x^{(2)})^2|+...+|(x^{(n)})^2-(y^{(n)})^2|+|(y^{(n)})^2\\
&-(z^{(n)})^2|+|(z^{(n)})^2-(x^{(n)})^2|\}\\
&=\frac{1}{n}\{G((x^{(1)})^2,(y^{(1)})^2),(z^{(1)})^2)+G((x^{(2)})^2,(y^{(2)})^2),(z^{(2)})^2)+...+G((x^{(n)})^2,(y^{(n)})^2),(z^{(n)})^2)\}\\
&-\frac{1}{2n}\{G((x^{(1)})^2,(y^{(1)})^2),(z^{(1)})^2)+G((x^{(2)})^2,(y^{(2)})^2),(z^{(2)})^2)+...+G((x^{(n)})^2,(y^{(n)})^2),(z^{(n)})^2)\}\\
&=\{\frac{G(gx^{(1)},gy^{(1)},gy^{(1)})+\dots+G(gx^{(n)},gy^{(n)},gy^{(n)})}{n}\}-\phi\{\frac{G(gx^{(1)},gy^{(1)},gy^{(1)})+\dots+G(gx^{(n)},gy^{(n)},gy^{(n)})}{n}\},
\end{aligned}$$

inequality holds.

Case 2. Let $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)},y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)},z^{(1)},z^{(2)},z^{(3)},\ldots,z^{(n)} \in X$ such that $x^{(i+1)} \preceq x^{(i)}$, $y^{(i+1)} \preceq y^{(i)}$, $z^{(i)} \preceq z^{(i+1)}$ for at least one i, then (for $z^{(1)} \preceq z^{(2)}$), $i = 1,3,...,n-1$. Then

$$
\begin{aligned}
&G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),F(y^{(1)},y^{(2)},\ldots,y^{(n)}),0)) = |F(x^{(1)},x^{(2)},\ldots,x^{(n)}) - F(y^{(1)},y^{(2)},\ldots,y^{(n)})|\\
&+|F(y^{(1)},y^{(2)},\ldots,y^{(n)}) - 0| + |0 - F(x^{(1)},x^{(2)},\ldots,x^{(n)})|\\
&= |\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n} - \frac{(y^{(1)})^2-(y^{(2)})^2+(y^{(3)})^3-,\ldots,-(y^{(n)})^2}{2n}|\\
&+|\frac{(y^{(1)})^2-(y^{(2)})^2+(y^{(3)})^3-,\ldots,-(y^{(n)})^2}{2n}| + |\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n}|\\
&\le |\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n} - \frac{(y^{(1)})^2-(y^{(2)})^2+(y^{(3)})^3-,\ldots,-(y^{(n)})^2}{2n}|\\
&+|\frac{(y^{(1)})^2-(y^{(2)})^2+(y^{(3)})^3-,\ldots,-(y^{(n)})^2+(z^{(2)})^2-(z^{(1)})^2}{2n}|\\
&+|\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2+(z^{(2)})^2-(z^{(1)})^2}{2n}|\\
&= |\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n} - \frac{(y^{(1)})^2-(y^{(2)})^2+(y^{(3)})^3-,\ldots,-(y^{(n)})^2}{2n}|\\
&+|\frac{(y^{(1)})^2-(z^{(1)})^2-((y^{(2)})^2-(z^{(2)})^2)+(y^{(3)})^3-,\ldots,-(y^{(n)})^2)}{2n}|\\
&+|\frac{(x^{(1)})^2-z^{(1)})^2-((x^{(2)})^2-z^{(2)})^2)+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n}|\\
&\le \frac{1}{2n}\{|(x^{(1)})^2-(y^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+\ldots+|(x^{(n)})^2-(y^{(n)})^2|+|(y^{(1)})^2-(z^{(1)})^2|+|(y^{(2)})^2\\
&-(z^{(2)})^2|+\ldots+|(y^{(n)})^2-(z^{(n)})^2|+|(z^{(1)})^2-(x^{(1)})^2|+|(z^{(2)})^2-(x^{(2)})^2|+\ldots+|(z^{(n)})^2-(x^{(n)})^2|\}\\
&= \frac{1}{n}\{|(x^{(1)})^2-(y^{(1)})^2|+|(y^{(1)})^2-(z^{(1)})^2|+(z^{(1)})^2-(x^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+|(y^{(2)})^2-(z^{(2)})^2|\\
&+|(z^{(2)})^2-(x^{(2)})^2|+\ldots+|(x^{(n)})^2-(y^{(n)})^2|+|(y^{(n)})^2-(z^{(n)})^2|+|(z^{(n)})^2-(x^{(n)})^2|\} - \frac{1}{2n}\{|(x^{(1)})^2\\
&-(y^{(1)})^2|+|(y^{(1)})^2-(z^{(1)})^2|+(z^{(1)})^2-(x^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+|(y^{(2)})^2-(z^{(2)})^2|+|(z^{(2)})^2-(x^{(2)})^2|\\
&+\ldots+|(x^{(n)})^2-(y^{(n)})^2|+|(y^{(n)})^2-(z^{(n)})^2|+|(z^{(n)})^2-(x^{(n)})^2|\}\\
&= \frac{1}{n}\{G((x^{(1)})^2,(y^{(1)})^2),(z^{(1)})^2)+G((x^{(2)})^2,(y^{(2)})^2),(z^{(2)})^2)+\ldots+G((x^{(n)})^2,(y^{(n)})^2),(z^{(n)})^2)\}\\
&-\frac{1}{2n}\{G((x^{(1)})^2,(y^{(1)})^2),(z^{(1)})^2)+G((x^{(2)})^2,(y^{(2)})^2),(z^{(2)})^2)+\ldots+G((x^{(n)})^2,(y^{(n)})^2),(z^{(n)})^2)\}\\
&= \{\frac{G(gx^{(1)},gy^{(1)},gy^{(1)})+\ldots+G(gx^{(n)},gy^{(n)},gy^{(n)})}{n}\} - \phi\{\frac{G(gx^{(1)},gy^{(1)},gy^{(1)})+\ldots+G(gx^{(n)},gy^{(n)},gy^{(n)})}{n}\}
\end{aligned}
$$

inequality holds.

Case 3. Let $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)},y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)},z^{(1)},z^{(2)},z^{(3)},\ldots,z^{(n)} \in X$ such that $x^{(i)} \preceq x^{(i+1)}$for at least one i, then (for $x^{(1)} \preceq x^{(2)}$), $y^{(i+1)} \preceq y^{(i)}$, $z^{(i+1)} \preceq z^{(i)}$ for $i = 1,3,...,n-1$. Similar manner of case 2 we can satisfy the inequality.

Case 4: Let $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)},y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)},z^{(1)},z^{(2)},z^{(3)},\ldots,z^{(n)} \in X$ such that $x^{(i+1)} \preceq x^{(i)}$, $y^{(i)} \preceq y^{(i+1)}$, for at least one i,Then (for $y^{(1)} \preceq y^{(2)}$), $z^{(i+1)} \preceq z^{(i)}$ for $i = 1,3,...,n-1$. Similar manner of case 2 we can satisfy the inequality.

Case 5. Let $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)},y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)},z^{(1)},z^{(2)},z^{(3)},\ldots,z^{(n)}\in X$ such that $x^{(i+1)}\preceq x^{(i)}$, $y^{(i)}\preceq y^{(i+1)}$, for at least one i,Then (for $y^{(1)}\preceq y^{(2)}$), $z^{(i)}\preceq z^{(i+1)}$for at least one i, then (for $z^{(1)}\preceq z^{(2)}$), for $i=1,3,...,n-1$.Then

$$
\begin{aligned}
&G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),F(y^{(1)},y^{(2)},\ldots,y^{(n)}),F(z^{(1)},z^{(2)},\ldots,z^{(n)}))=G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),0,0)\\
&=|\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n}|+|\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n}|\\
&\leq|\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2+(y^{(2)})^2-(y^{(1)})^2}{2n}|\\
&+|\frac{(x^{(1)})^2-(x^{(2)})^2+(x^{(3)})^3-,\ldots,-(x^{(n)})^2+(z^{(2)})^2-(z^{(1)})^2}{2n}|\\
&=|\frac{(x^{(1)})^2-(y^{(1)})^2-((x^{(2)})^2-(y^{(2)})^2)+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n}|\\
&+|\frac{(x^{(1)})^2-(z^{(1)})^2-((x^{(2)})^2-(z^{(2)})^2)+(x^{(3)})^3-,\ldots,-(x^{(n)})^2}{2n}|\\
&\leq\frac{|(x^{(1)})^2-(y^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+,\ldots,+|(x^{(n)})^2-y^{(n)})^2|}{2n}\\
&+\frac{|(z^{(1)})^2-(x^{(1)})^2|+|(z^{(2)})^2-(x^{(2)})^2|+,\ldots,+|(z^{(n)})^2-x^{(n)})^2|}{2n}\\
&\leq\frac{|(x^{(1)})^2-(y^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+,\ldots,+|(x^{(n)})^2-(y^{(n)})^2|}{2n}\\
&+\frac{|(z^{(1)})^2-(x^{(1)})^2|+|(z^{(2)})^2-(x^{(2)})^2|+,\ldots,+|(z^{(n)})^2-(x^{(n)})^2|}{2n}\\
&+|\frac{(z^{(2)})^2-(z^{(1)})^2-((y^{(2)})^2-(y^{(1)})^2)}{2n}|\\
&\leq\frac{|(x^{(1)})^2-(y^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+,\ldots,+|(x^{(n)})^2-(y^{(n)})^2|}{2n}\\
&+\frac{|(z^{(1)})^2-(x^{(1)})^2|+|(z^{(2)})^2-(x^{(2)})^2|+,\ldots,+|(z^{(n)})^2-(x^{(n)})^2|}{2n}\\
&+\frac{|(y^{(1)})^2-(z^{(1)})^2|+|(y^{(2)})^2-(z^{(2)})^2|+,\ldots,+|(y^{(n)})^2-(z^{(n)})^2|}{2n}\\
&=\frac{1}{n}\{|(x^{(1)})^2-(y^{(1)})^2|+|(y^{(1)})^2-(z^{(1)})^2|+(z^{(1)})^2-(x^{(1)})^2|\\
&+|(x^{(2)})^2-(y^{(2)})^2|+|(y^{(2)})^2-(z^{(2)})^2|+...+|(z^{(n)})^2-(x^{(n)})^2|+|(x^{(n)})^2-(y^{(n)})^2|\\
&+|(y^{(n)})^2-(z^{(n)})^2|+|(z^{(n)})^2-(x^{(n)})^2|-\frac{1}{2n}\{|(x^{(1)})^2-(y^{(1)})^2|\\
&+|(y^{(1)})^2-(z^{(1)})^2|+(z^{(1)})^2-(x^{(1)})^2|+|(x^{(2)})^2-(y^{(2)})^2|+|(y^{(2)})^2-(z^{(2)})^2|+...+|(z^{(n)})^2-(x^{(n)})^2|\\
&+|(x^{(n)})^2-(y^{(n)})^2|+|(y^{(n)})^2-(z^{(n)})^2|+|(z^{(n)})^2-(x^{(n)})^2\}\\
&=\frac{1}{n}\{G((x^{(1)})^2,(y^{(1)})^2),(z^{(1)})^2)+...+G((x^{(n)})^2,(y^{(n)})^2),(z^{(n)})^2)\}\\
&-\frac{1}{2n}\{G((x^{(1)})^2,(y^{(1)})^2),(z^{(1)})^2)+...+G((x^{(n)})^2,(y^{(n)})^2),(z^{(n)})^2)\}\\
&=\{\frac{G(gx^{(1)},gy^{(1)},gy^{(1)})+\ldots+G(gx^{(n)},gy^{(n)},gy^{(n)})}{n}\}-\phi\{\frac{G(gx^{(1)},gy^{(1)},gy^{(1)})+\ldots+G(gx^{(n)},gy^{(n)},gy^{(n)})}{n}\}
\end{aligned}
$$

inequality holds.

Case 6. Let $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)},y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)},z^{(1)},z^{(2)},z^{(3)},\ldots,z^{(n)}\in X$ such that $x^{(i)}\preceq x^{(i+1)}$, for at least one

i, Then (for $x^{(1)} \preceq x^{(2)}$), $y^{(i+1)} \preceq y^{(i)}$, $z^{(i)} \preceq z^{(i+1)}$ for at least one i, then (for $z^{(1)} \preceq z^{(2)}$), for $i = 1,3,...,n-1$. Similar manner of case 5 we can satisfy inequality.

Case 7. Let $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)},y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)},z^{(1)},z^{(2)},z^{(3)},\ldots,z^{(n)} \in X$ such that $x^{(i)} \preceq x^{(i+1)}$, for at least one i, Then (for $x^{(1)} \preceq x^{(2)}$), $y^{(i)} \preceq y^{(i+1)}$, for at least one i, then (for $y^{(1)} \preceq y^{(2)}$), $z^{(i+1)} \preceq z^{(i)}$ for $i = 1,3,...,n-1$. Similar manner of case 5 we can satisfy inequality.

Case 8. Let $x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)},y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)},z^{(1)},z^{(2)},z^{(3)},\ldots,z^{(n)} \in X$ such that $x^{(i)} \preceq x^{(i+1)}$, $y^{(i)} \preceq y^{(i+1)}$, $z^{(i)} \preceq z^{(i+1)}$for at least one i. Then

$$G(F(x^{(1)},x^{(2)},\ldots,x^{(n)}),F(y^{(1)},y^{(2)},\ldots,y^{(n)}),F(z^{(1)},z^{(2)},\ldots,^{(n)})) = G(0,0,0)$$
$$\leq \{\frac{G(gx^{(1)},gy^{(1)},gz^{(1)})+\ldots+G(gx^{(n)},gy^{(n)},gz^{(n)})}{n}\} - \phi\{\frac{G(gx^{(1)},gy^{(1)},gz^{(1)})+\ldots+G(gx^{(n)},gy^{(n)},gz^{(n)})}{n}\}$$

inequality holds. Hence all the condition of the Theorem 3.1 are satisfied and $(0,0,0,...,0)$ is an n-tupled coincidence point of F and g.

5 Acknowledgement

The first author PPM is thankful to University Grants Commission, New Delhi, India for financial assistance through Major Research Project File number 42-32/2013 (SR).

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