New Trends in Mathematical Sciences
http://www.ntmsci.com

The k-step α-generalized Pell-Padovan sequence in Finite Groups

Sait Tas[1], *Omur Deveci*[2] *and Erdal Karaduman*[1]

[1]Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, Turkey
[2]Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey

Received: 13 April 2015, Revised: 31 May 2015, Accepted: 25 August 2015
Published online: 9 November 2015

Abstract: In this paper, we study the k-step α-generalized Pell-Padovan sequence modulo m. We define the k-step α-generalized Pell-Padovan sequences in a finite group and we examine the periods of these sequences. Also, we obtain the periods of the k-step α-generalized Pell-Padovan sequences in the semidihedral group SD_{2^m}.

Keywords: Sequence, period, matrix, group.

1 Introduction and Preliminaries

The Pell-Padovan sequence $\{P(n)\}$ is defined [12,13] recursively by the equation

$$P(n+3) = 2P(n+1) + P(n) \tag{1}$$

for $n \geq 0$, where $P(0) = P(1) = P(2) = 1$.

Kalman [8] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \cdots, c_{k-1}$ are real constants. In [8], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = [a_{ij}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}_{k\times k}.$$

* Corresponding author e-mail: saittas@atauni.edu.tr

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence.

The study of the linear recurrence sequences in groups began with the earlier work of Wall [14] where the ordinary Fibonacci sequences in cyclic groups were investigated. Recently, many authors have studied on some special linear recurrence sequences in groups; see for example, [1-7,9-11]. Now we extend the concept to the k-step α-generalized Pell-Padovan sequences. In this paper, the usual notation p is used for a prime number.

2 The k-step α-generalized Pell-Padovan sequence

The k-step α-generalized Pell-Padovan sequence is defined as

$$P_k^\alpha(n+k+1) = 2^\alpha P_k^\alpha(n+k-1) + P_k^\alpha(n+k-2) + \cdots + P_k^\alpha(n) \tag{2}$$

for $n \geq 0$, where $P_k^\alpha(0) = P_k^\alpha(1) = \cdots = P_k^\alpha(k) = 1$.

When $k = 2$ and $\alpha = 1$, this sequence reduces to the usual Pell-Padovan sequence, $\{P(n)\}$. By (2), we have

$$\begin{bmatrix} P(n) \\ P(n+1) \\ P(n+2) \\ \vdots \\ P(n+k) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 2^\alpha & 0 \end{bmatrix} \begin{bmatrix} P(n-1) \\ P(n) \\ P(n+1) \\ \vdots \\ P(n+k-1) \end{bmatrix}$$

for the k-step α-generalized Pell-Padovan sequence. Let

$$M = [m_{ij}]_{(k+1)\times(k+1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 2^\alpha & 0 \end{bmatrix}_{(k+1)\times(k+1)}.$$

By inductive argument we have

$$M^{n}\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}=\begin{bmatrix}P(n)\\P(n+1)\\P(n+2)\\\vdots\\P(n+k)\end{bmatrix} \tag{3}$$

for $n \geq 0$.

3 The k-step α-generalized Pell-Padovan sequence modulo m

Reducing the k-step α-generalized Pell-Padovan sequence modulo m, we can get a repeating sequence, denoted by

$$\left\{P_{k,m}^{\alpha}(n)\right\}=\left\{P_{k,m}^{\alpha}(0),\,P_{k,m}^{\alpha}(1),\,P_{k,m}^{\alpha}(2),\cdots,P_{k,m}^{\alpha}(k),\cdots\right\}$$

where $P_{k,m}^{\alpha}(i) \equiv P_{k}^{\alpha}(i) \pmod{m}$ and it has the same recurrence relation as in (2).

Theorem 3.1. $\left\{P_{k,m}^{\alpha}(n)\right\}$ is a simply periodic sequence.

Proof. Let $A_{k+1}=\{(a_0,a_1,\cdots,a_k) | 0 \leq a_i \leq m-1\}$. Then we have $S(A_{k+1})=m^{k+1}$ being finite ($S(A_{k+1})$ mean that the number of the elements of A_{k+1}) , that is, for any $v \geq 0$, there exist $w \geq v+k$ such that

$$P_{k,m}^{\alpha}(v+1)=P_{k,m}^{\alpha}(w+1),\,P_{k,m}^{\alpha}(v+2)=P_{k,m}^{\alpha}(w+2),\cdots,P_{k,m}^{\alpha}(v+p+1)=P_{k,m}^{\alpha}(w+p+1).$$

From definition of the the k-step α-generalized Pell-Padovan sequence we have $P_{k}^{\alpha}(n) = P_{k}^{\alpha}(n+k+1) - 2^{\alpha}P_{k}^{\alpha}(n+k-1) - P_{k}^{\alpha}(n+k-2) - \cdots - P_{k}^{\alpha}(n+1)$ so if $P_{k,m}^{\alpha}(v) = P_{k,m}^{\alpha}(w)$, $P_{k,m}^{\alpha}(v-1)=P_{k,m}^{\alpha}(w-1)$, $\cdots$, $P_{k,m}^{\alpha}(1)=P_{k,m}^{\alpha}(w-v+1)$ and $P_{k,m}^{\alpha}(0)=P_{k,m}^{\alpha}(w-v)$, which implies that this sequence is simply periodic. We denote the smallest period of $\left\{P_{k,m}^{\alpha}(n)\right\}$ by $hP_{k,m}^{\alpha}$.

Example 1. We have $\left\{P_{5,2}^{1}(n)\right\}=\{1,1,1,1,1,1,0,0,0,1,0,1,1,1,0,1,1,1,1,1,1,\cdots\}$, and then repeat. So, we get $hP_{5,2}^{1}=15$.

For given a matrix $M=[e_{ij}]_{(k+1)\times(k+1)}$ with e_{ij}'s being integers, $M \pmod{m}$ means that every entries of M are reduced modulo m, that is, $M\ (mod\ m) \equiv (e_{ij}\ (mod\ m))$. Let a be an positive integer and let $\langle M\rangle_{p^a}=\left\{M^{i} \pmod{p^{a}} \middle| i \geq 0\right\}$. Then, it is clear that the set $\langle M\rangle_{p^a}$ is a cyclic group. Let $\left|\langle M\rangle_{p^a}\right|$ denote the order of $\langle M\rangle_{p^a}$.

Let a be an positive integer, then by (3), it is shown that $hP_{k,p^a}^{\alpha}=\left|\langle M\rangle_{p^a}\right|$.

Theorem 3.2. Let t be the largest positive integer such that $hP_{k,p}^{\alpha}=hP_{k,p^t}^{\alpha}$. Then $hP_{k,p^u}^{\alpha}=p^{u-t}hP_{k,p}^{\alpha}$, for every $u \geq t$.

Proof. By $hP_{k,p^a}^{\alpha}=\left|\langle M\rangle_{p^a}\right|$ we see that for each positive integer λ, $M^{hP_{k,p^{\lambda+1}}^{\alpha}} \equiv I \pmod{p^{\lambda+1}}$, hence,

$M^{hP^{\alpha}_{k,p^{\lambda+1}}} \equiv I \left(\bmod p^{\lambda}\right)$, which means that $hP^{\alpha}_{k,p^{\lambda}}$ divides $hP^{\alpha}_{k,p^{\lambda+1}}$. Also, we can write $M^{hP^{\alpha}_{k,p^{\lambda}}} = I + \left(m^{(\lambda)}_{ij} p^{\lambda}\right)$. Thus,

$$M^{\left(hP^{\alpha}_{k,p^{\lambda}}\right)p} = \left(I + \left(m^{(\lambda)}_{ij} p^{\lambda}\right)\right)^{p} = \sum_{i=0}^{p} \binom{p}{i} \left(m^{(\lambda)}_{ij} p^{\lambda}\right)^{i} \equiv I \left(\bmod p^{\lambda+1}\right),$$

which yields that $hP^{\alpha}_{k,p^{\lambda+1}}$ divides $\left(hP^{\alpha}_{k,p^{\lambda+1}}\right)p$. So, we can write $hP^{\alpha}_{k,p^{\lambda+1}} = hP^{\alpha}_{k,p^{\lambda}}$ or $hP^{\alpha}_{k,p^{\lambda+1}} = \left(hP^{\alpha}_{k,p^{\lambda}}\right)p$, and the latter holds if, and only if, there is a $m^{(\lambda)}_{ij}$ which is not divisible by p. Since $hP^{\alpha}_{k,p^{t}} \neq hP^{\alpha}_{k,p^{t+1}}$, there is an $m^{(t+1)}_{ij}$ which is not divisible by p, thus, $hP^{\alpha}_{k,p^{t+1}} \neq hP^{\alpha}_{k,p^{t+2}}$. The proof is completed by induction on t.

Theorem 3.3. If $m = \prod_{i=1}^{u} p_i^{v_i}$, $(u \geq 1)$ where p_i's are distinct primes, then $hP^{\alpha}_{k,p^{m}}$ =lcm$\left[hP^{\alpha}_{k,p_1^{v_1}}, hP^{\alpha}_{k,p_2^{v_2}}, \cdots, hP^{\alpha}_{k,p_u^{v_u}}\right]$.

Proof. Since $hP^{\alpha}_{k,p_i^{v_i}}$ is the period of $\left\{P^{\alpha}_{k,p_i^{v_i}}(n)\right\}$, the sequence $\left\{P^{\alpha}_{k,p_i^{v_i}}(n)\right\}$ repeats only after blocks of length $\beta.hP^{\alpha}_{k,p_i^{v_i}}$, ($\beta$ is a natural numbers). Also, since $hP^{\alpha}_{k,m}$ is the period $\left\{P^{\alpha}_{k,m}(n)\right\}$, the sequence $\left\{P^{\alpha}_{k,p_i^{v_i}}(n)\right\}$ repeats after $hP^{\alpha}_{k,m}$ terms for all values i. Thus, $hP^{\alpha}_{k,m}$ is of the form $\beta.hP^{\alpha}_{k,p_i^{v_i}}$ for all values of i, and since any such number gives a period of $\left\{P^{\alpha}_{k,p_i^{v_i}}(n)\right\}$. So we get $hP^{\alpha}_{k,p^{m}}$ =lcm$\left[hP^{\alpha}_{k,p_1^{v_1}}, hP^{\alpha}_{k,p_2^{v_2}}, \cdots, hP^{\alpha}_{k,p_u^{v_u}}\right]$.

4 The k-step α-generalized Pell-Padovan sequence in groups

Definition 4.1. For a generating pair $(x,y) \in G$, we define the Pell-Padovan orbit $P_{x,y,y}(G) = \{x_i\}$ and co-Pell-Padovan orbit $c - P_{x,y,y}(G) = \{x_i\}$, respectively as follows:

$$x_0 = xy,\ x_1 = y,\ x_2 = y,\ x_{i+2} = (x_{i-1}) \cdot (x_i)^2,\ \ i \geq 1$$

and

$$x_0 = yx,\ x_1 = y,\ x_2 = y,\ x_{i+2} = (x_{i-1}) \cdot (x_i)^2,\ \ i \geq 1.$$

Definition 4.2. A k-step α-generalized Pell-Padovan sequence in a finite group is a sequence of group elements $a_0, a_1, \cdots a_n, \cdots$ for which, given an initial (seed) set $a_0 = x_0, a_1 = x_1, \cdots, a_{j-1} = x_{j-1}, a_j = x_{j-1}$, each element is defined by

$$a_n = \begin{cases} a_0 a_1 \cdots (a_{n-2})^2 & \text{for } j < n \leq k, \\ a_{n-k-1} \cdots (a_{n-2})^2 & \text{for } n > k. \end{cases}$$

It is require that the initial elements of the sequence, $x_0, \cdots, x_{j-1}$, generate the group, thus, forcing the k-step α-generalized Pell-Padovan sequence to reflect the structure of the group. We denote the k-step α-generalized Pell-Padovan sequence of a group generated by $x_0, \cdots, x_{j-1}$ by $PP_k\left(G; x_0, \cdots, x_{j-1}\right)$.

The k-step α-generalized Pell-Padovan sequence in a cyclic group C_n of order n can be written as $PP_k\left(C_n; x, x, \cdots, x\right)$.

Theorem 4.1. A k-step α-generalized Pell-Padovan sequence in a finite group is periodic.

Proof. Let G be a finite group and $|G|$ be the order of G. Since there are $|G|^{k+1}$ distinct k+1-tuples of elements of the group G, at least one of the k+1-tuples appears twice in a k-step α-generalized Pell-Padovan sequence of the group G. Because of the repeating, the k-step α-generalized Pell-Padovan sequence is periodic.

We denote the period of the sequence $PP_k\left(G;x_0,\cdots,x_{j-1}\right)$ by $Per_k\left(G;x_0,\cdots,x_{j-1}\right)$. From the definition it is clear that the period of the k-step α-generalized Pell-Padovan sequence in a finite group depends on the chosen generating set and the order in which the assignments of $x_0,x_1,x_2,\cdots,x_{j-1}$.

It is clear that $hP^1_{k,n} = Per_k\left(C_n;x,x,\cdots,x\right)$.

A group SD_{2^m} is semidihedral group of order 2^m if

$$SD_{2^m}=\left\langle x,y|x^{2^{m-1}}=y^2=e,\ yxy=x^{-1+2^{m-2}}\right\rangle$$

for every $m\geq 4$. Note that the orders x and y are 2^{m-1} and 2, respectively.

Theorem 4.2. The periods of the k-step α-generalized Pell-Padovan sequences in the semidihedral group SD_{2^m} for initial (seed) set $x,\ y$ are as follows:

i. $Per_2\left(SD_{2^m};x,\ y\right)=3\cdot 2^{m-2}$

ii. $Per_k\left(SD_{2^m};x,\ y\right)=hP^1_{k,2}\cdot 2^{m-2}$ for $3\leq k\leq 4$.

iii. $Per_k\left(SD_{2^m};x,\ y\right)=hP^1_{k,2}\cdot 2^{m-3}$ for $k\geq 5$.

Proof.i. The sequence $PP_2\left(SD_{2^m};x,\ y\right)$ is

$$x,\ y,\ y,\ x,\ y,\ x^{2^{m-1}-2}y,\ x,\ y,\ x^{2^{m-1}-4}y,\ x,\ y,\ x^{2^{m-1}-6}y,\cdots.$$

Using the relations of the SD_{2^m}, this sequence becomes:

$$\begin{aligned}&x_0=x,\ x_1=y,\ x_2=y,\ \cdots,\\&x_{3i}=x,\ x_{3i+1}=y,\ x_{3i+2}=x^{2^{m-1}-2i}y,\ \cdots.\end{aligned}$$

So we need the smallest such that $2^{m-1}-2i=0$, i is a natural numbers. Thus, we obtain $x_{3\cdot 2^{m-2}}=x$, $x_{3\cdot 2^{m-2}+1}=y$ and $x_{3\cdot 2^{m-2}+2}=y$. Since the elements succeeding $x_{3\cdot 2^{m-2}},x_{3\cdot 2^{m-2}+1},x_{3\cdot 2^{m-2}+2}$, depend on x,y and y for their values, the cycle begins again with the $\left(3\cdot 2^{m-2}\right)^{\text{nd}}$ element. So we get $Per_2\left(SD_{2^m};x,\ y\right)=3\cdot 2^{m-2}$.

ii. Note that $hP^1_{3,2}=15$ and the sequence $PP_3\left(SD_{2^m};x,\ y\right)$ is

$$\begin{aligned}&x,\ y,\ y,\ x,\ xy,\ x^2,\ x^{2^{m-1}-1}y,\ x^{2^{m-1}-2}y,\ x^{2^{m-2}-1}y,\ xy,\\&x^{2^{m-2}+1},\ x^{2^{m-2}-1},\ e,\ x^2y,\ e,\ x^{2^{m-2}-1},\ x^2y,\ x^4y,\ x^{2^{m-2}-1},\cdots.\end{aligned}$$

So, by the relations of the SD_{2^m}, this sequence becomes:

$$\begin{aligned}
&x_0 = x,\ x_1 = y,\ x_2 = y,\ x_3 = x,\ \cdots,\\
&x_{30} = x,\ x_{31} = x^4y,\ x_{32} = y,\ x_{33} = x^{2^{m-1}-3},\ \cdots,\\
&x_{30i} = x,\ x_{30i+1} = x^{4i}y,\ x_{30i+2} = y,\ x_{30i+3} = x^{2^{m-1}-4i+1},\ \cdots.
\end{aligned}$$

So we need the smallest such that $2^{m-1} = 4i$, i is a natural numbers. If $2^{m-3} = i$, we obtain $x_{30\cdot 2^{m-3}} = x$, $x_{30\cdot 2^{m-3}+1} = y$, $x_{30\cdot 2^{m-2}+2} = y$ and $x_{30\cdot 2^{m-2}+3} = x$. Since the elements succeeding $x_{15\cdot 2^{m-2}}, x_{15\cdot 2^{m-2}+1}, x_{15\cdot 2^{m-2}+2}, x_{15\cdot 2^{m-2}+3}$ depend on x, y, y and x for their values, the cycle begins again with the $\left(15\cdot 2^{m-2}\right)^{\text{nd}}$ element. So we get $Per_3\left(SD_{2^m}; x,\ y\right) = 15\cdot 2^{m-2} = hP^1_{3,2}2^{m-2}$.

The proof for $k = 4$ is similar to the above and it is omitted.

iii. Let $k \geq 5$. We have the sequence

$$\begin{aligned}
&x_0 = x,\ x_1 = y,\ x_2 = y,\ x_3 = x,\ x_4 = xy,\ x_5 = x^3,\ \cdots,\\
&x_{2hP^1_{k,2}-k+3} = e,\ x_{2hP^1_{k,2}-k+2} = e,\ \cdots,\ x_{2hP^1_{k,2}-1} = e,\\
&x_{2hP^1_{k,2}} = x^9,\ x_{2hP^1_{k,2}+1} = y,\ x_{2hP^1_{k,2}+2} = y,\ x_{2hP^1_{k,2}+3} = x,\ x_{2hP^1_{k,2}+4} = x^9y,\ x_{2hP^1_{k,2}+5} = x^{11},\ \cdots,\\
&x_{2hP^1_{k,2}\cdot i-k+3} = e,\ x_{2hP^1_{k,2}\cdot i-k+2} = e,\ \cdots,\ x_{2hP^1_{k,2}\cdot i-1} = e,\\
&x_{2hP^1_{k,2}\cdot i} = x^{8i+1},\ x_{2hP^1_{k,2}\cdot i+1} = y,\ x_{2hP^1_{k,2}\cdot i+2} = y,\ x_{2hP^1_{k,2}\cdot i+3} = x,\ x_{i\cdot 2hP^1_{k,2}+4} = x^{8i+1}y,\ x_{2hP^1_{k,2}\cdot i+5} = x^{8i+3},\ \cdots.
\end{aligned}$$

So, we need the smallest such that $2^{m-1} = 8i$, i is a natural numbers. If $2^{m-4} = i$, we obtain $x_{hP^1_{k,2}\cdot 2^{m-3}-k+3} = e$, $x_{hP^1_{k,2}\cdot 2^{m-3}-k+2} = e$, $\cdots$, $x_{hP^1_{k,2}\cdot 2^{m-3}-1} = e$, $x_{hP^1_{k,2}\cdot 2^{m-3}} = x$, $x_{hP^1_{k,2}\cdot 2^{m-3}+1} = y$, $x_{hP^1_{k,2}\cdot 2^{m-3}+2} = y$, $x_{hP^1_{k,2}\cdot 2^{m-3}+3} = x$, $x_{hP^1_{k,2}\cdot 2^{m-3}+4} = xy$ and $x_{hP^1_{k,2}\cdot 2^{m-3}+5} = x^3$. Thus, the cycle begins again with the $\left(hP^1_{k,2}\cdot 2^{m-3}\right)^{\text{nd}}$ element. So we get $Per_k\left(SD_{2^m}; x,\ y\right) = hP^1_{k,2}\cdot 2^{m-3}$.

5 Acknowledgment

This Project was supported by the Commission for the Scientific Research Projects of Kafkas University. The Project number. 2013-FEF-72.

References

[1] H. Aydın, G. C. Smith, Finite p-quotients of some cyclically presented groups, J. London Math. Soc. 49.2 (1994) 83-92.

[2] C. M. Campbell, H. Doostie and E. F. Robertson, Fibonacci length of generating pairs in groups in Applications of Fibonacci Numbers, Vol. 3 Eds. G. E. Bergum et al. Kluwer Academic Publishers, (1990), 27-35.

[3] O. Deveci and E. Karaduman, The generalized order-k Lucas sequences in Finite groups, J. Appl. Math., 464580-1-464580-15 (2012).

[4] O. Deveci, M. Akdeniz and E. Karaduman, The generalized Pell p-sequences in groups", Ars Combinatoria, in press..

[5] O. Deveci and E. Karaduman, The Pell sequences in finite groups, Util. Math., in press.

[6] O. Deveci, The Pell-Padovan sequences and the Jacobsthal-Padovan sequences in finite groups, Util. Math., in press.

[7] H. Doostie and M. Hashemi, Fibonacci lengths involving the Wall number $k(n)$, J. Appl. Math. Comput. 20 (2006), 171-180.

[8] D. Kalman, Generalized Fibonacci numbers by matrix methods, The Fibonacci Quart., 20, (1982), 73-76.

[9] S.W. Knox, Fibonacci sequences in finite groups, The Fibonacci Quart., 30, (1992), 116-120.

[10] K. Lü and J. Wang, *k*-step Fibonacci sequence modulo *m*, Util. Math., 71 (2007), 169-178.

[11] E. Ozkan, H. Aydin and R. Dikici, 3-step Fibonacci series modulo *m*, Appl. Math. and Compt., 143 (2003), 165-172.

[12] A.G. Shannon, P.G. Anderson and A.F. Horadam, Properties of cordonnieri Perrin and Van der Lann numbers, International Journal of Mathematical Education in Science & Tecnology, 37,(2006), 825-831.

[13] A.G. Shannon, A.F. Horadam and P.G. Anderson, The auxiliary equation associated with plastic number, Notes on Number Theory and Discrete Mathematics, 12, (2006), 1-12.

[14] D.D. Wall, Fibonacci series modulo *m*, Amer. Math. Monthly, 67 (1960), 525-532.