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Applications of the two-dimensional differential transform and least square method for solving nonlinear wave equations

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Abstract: The differential transform and least square are analytical methods for solving differential equations. In this article, two-Dimensional Differential Transform Method (2D DTM) and Least Square Method (LSM) are applied to obtaining the analytic solution of the two- dimensional non- linear wave equations. We demonstrate that the differential transform method and least square are very effective and convenient for achieving the analytical solutions of linear or nonlinear partial differential equations. Also, three examples are given to demonstrate the exactness of the methods. Results of these methods are compared with the exact solution.

Keywords: Two-Dimensional Transform Method (2D DTM), wave equation, analytical solution, Least Square Method (LSM) and exact solution

1. Introduction

Recently, analytical method is widely used for solving linear and nonlinear equations. The Differential Transformation Method (DTM) is one of the analytical methods employed in this article which was firstly applied in the engineering gamut by Zhou in 1986 for solving linear and nonlinear equation [1]. This method achieves a solution based on Taylor series. The differential transform method has been developed for solving the differential equations. For example, in [2-4] this method applied to partial differential equations, in [5-7] one-dimensional Volterra integral and integro-differential equations solved by differential transform method and in [8-18] this method applied to linear and nonlinear equations.

The Weighted Residuals Methods (WRMs) are accurate approximation techniques for solving differential equations. Least Square Method (LSM) which is one of the weighted residuals methods that recently introduced by Aziz and Bouaziz[19, 20] and is applied for predicting the performance of a longitudinal fin. Hatami and Ganji[21] applied and analyzed least square method for micro channel heat sink cooled by Cu-water Nano fluid.

In the present work, we employed two dimensional differential transform method and least square method for solving nonlinear two- dimensional wave equations and the results are compared with the exact solution.

2. Method of solution

2.1. Two- dimensional differential transformation method

In this section, the fundamental idea of two- differential transform method (2D DTM) is concisely introduced [22-25].

We assume a function $u(t,x)$ that is analytic and differentiated continuously with respect to time t in the domain of interest, then let

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}w(t,x)}{\partial t^k \partial x^s}\right]_{(t,x)=(0,0)}, \tag{1}$$

Where $w(t,x)$ is the origin function and $W(k,h)$ is transform function. The transformation is called the T- function. The differential inverse transform of $W(k,h)$ is defined as follows

$$w(t,x)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)t^k x^h, \tag{2}$$

And from Eq.

(1)and Eq. (2)can be concluded

$$w(t,x)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}w(t,x)}{\partial t^k \partial x^h}\right]_{(t,x)=(0,0)} t^k x^h \tag{3}$$

And Eq. (2)can be rewritten as

$$w(t,x)=\sum_{k=0}^{m}\sum_{h=0}^{n} W(k,h)t^k x^h. \tag{4}$$

Eq. (4)implies that

$$w(t,x)=\sum_{k=m+1}^{\infty}\sum_{h=n+1}^{\infty} W(k,h)t^k x^h, \tag{5}$$

Some of the fundamental functions and transformed functions are listed in **Error! Reference source not found.**.

Origin function	**Transformed function**
$w(t,x)=u(t,x)\pm v(t,x)$	$W(k,h)=U(k,h)\pm V(k,h)$
$w(t,x)=\frac{\partial w(t,x)}{dt}$	$W(k,h)=(k+1)\ W(k+1,h)$
$w(t,x)=\frac{\partial w(t,x)}{dx}$	$W(k,h)=(h+1)\ W(k,h+1)$
$w(t,x)=\frac{\partial^{r+q}w(t,x)}{\partial t^r \partial x^q}$	$W(k,h)=(k+1)(k+2)\cdots(k+r)(h+1)(h+2)\cdots(h+q).\ W(k+r,h+q)$
$w(t,x)=u(t,x)v(t,x)$	$W(k,h)=\sum_{i=0}^{k}\sum_{j=0}^{h}U(i,h-j)\ V(k-i,j)$
$w(t,x)=\alpha u(t,x)$	$W(k,h)=\alpha\ U(k,h)$
$w(t,x)=t^m x^n$	$W(t,x)=\delta(k-m)\delta(h-n)$
$w(t,x)=\frac{\partial u(t,x)}{dt}\frac{\partial v(x,t)}{dx}$	$W(t,x)=\sum_{i=0}^{k}\sum_{j=0}^{h}(i+1)(k-i+1)\ U(i+1,s-j)\ V(k-i+1,j)$
$w(t,x)=x^m e^{(a\,t)}$	$W(k,h)=\frac{a^h}{h!}\delta(k-m)$
$w(t,x)=x^m\sin(at+b)$	$W(t,x)=\frac{a^h}{h!}\delta(k-m)\sin\left(\frac{h\pi}{2}+b\right)$

$w(t,x) = x^m \cos(at+b)$	$W(t,x) = \frac{a^h}{h!}\delta(k-m)\cos\left(\frac{h\pi}{2}+b\right)$

2.2. Least square method

Suppose a differential operator D, is acted on a function v to produce a function p

$$D(v(x)) = p(x) \tag{6}$$

It is considered that v is approximated by a function $\tilde{v}$, which is a linear combination of basic function chosen from a linearly independent set. That is

$$v \cong \tilde{v} = \sum_{j=1}^{n} c_j \varphi_j \tag{7}$$

Now, when substituted into the differential operator D, the result of the operations generally is not $p(x)$. Hence an error or residual will exist:

$$R(x) = D(\tilde{v}(x)) - p(x) \neq 0 \tag{8}$$

The main idea of the LSM is to force the residual to zero in some average sense over the domain. That is:

$$\int_X R(x) W_j(x) = 0, \qquad j = 1, 2, \ldots \mathrm{m} \tag{9}$$

Where the number of weight function W_j is exactly equal to the number of unknown constant c_j in. If the continuous summation of all the squared residuals is minimized, the rationale behind the same can be seen. In other words, a minimum of

$$S = \int_X R(x)R(x)dx = \int_X R^2(x)dx \tag{10}$$

In order to achieve a minimum of this scalar function, the derivatives of S with respect to all the unknown parameters must be zero. That is,

$$\frac{\partial S}{\partial c_j} = 2\int_X R(x)\frac{\partial R}{\partial c_j}dx = 0 \tag{11}$$

Comparing with Eq. (10), the weight functions are seen to be

$$W_j = 2\frac{\partial R}{\partial c_j} \tag{12}$$

However, the "2" coefficient can be dropped, since it cancels out in the equation. Therefore the weight functions for the least square method are just the derivatives of the residual with respect to the unknown constant

$$W_j = \frac{\partial R}{\partial c_j} \tag{13}$$

And for two- dimensional differential equation, Eq. (9), would be

$$\int_Y \int_X R(y,x) W_j(y,x) = 0, \quad j = 1, 2, \ldots m \tag{14}$$

3. Examples and methods applications

Consider the following two- dimensional nonlinear wave equation, with the represented initial conditions [26]:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - u\dfrac{\partial^2 u}{\partial t^2} = \phi(t,x), \quad 0 \le x,t \le 1, \\ u(t,0) = f(t), \quad \dfrac{\partial}{\partial x}u(t,0) = g(t) \ . \end{cases} \tag{15}$$

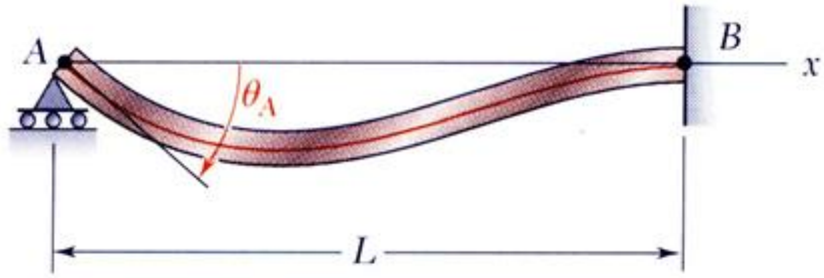

Fig.1. wave of beam

Fig.1depicts an example of wave of beam.

3.1. Example 1

Consider two- dimensional nonlinear wave equation with the initial condition are given by [26]

$$\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 u}{\partial t^2} = 1 - \frac{x^2 + t^2}{2}, \quad 0 \le \mathrm{x,t} \le 1, \tag{16}$$

$$\mathrm{u(t,0)} = \frac{\mathrm{t}^2}{2} \ , \quad \frac{\partial}{\partial x}\mathrm{u(t,0)=0} \ . \tag{17}$$

The exact solution of this example is given by [27]

$$u(t,x) = \frac{x^2 + t^2}{2}. \tag{18}$$

3.1.1. Differential transformation method

Taking two- dimensional differential transform to Eq. (16), we have

$$\begin{aligned} (h+1)(h+2).U(k,h+2) &= \sum_{s=0}^{h}\left(\sum_{r=0}^{k} U(r,h-s).(k-r+1)(k-r+2)U(k-r+2,s)\right) + \\ &\delta(h-0).\delta(k-0) - \frac{1}{2}\delta(h-2).\delta(k-0) - \frac{1}{2}\delta(k-2).\delta(h-0), \quad k,h = 0,1,2,\ldots,N. \end{aligned} \tag{19}$$

where $U(k,h)$ is the differential transform of $u(t,x)$.

With transformed initial conditions Eq.(17), would be

$$U(2,0)=\frac{1}{2} \quad , \quad U(i,0)=0 \quad , \ i=0,1,3,4,\ldots,m, \tag{20}$$

$$U(i,1)=0 \qquad\qquad , \ i=0,1,2,3,\ldots,n \tag{21}$$

Substituting Eq. (20) and Eq. (21) into Eq. (19), and by recursive method, we obtain the following

$$U(0,2)=\frac{1}{2} \tag{22}$$

And the others of $U(k,h)$ are zero.

Substituting all $U(k,h)$ into Eq. (5), we obtain the series solution as follows

$$u(t,x)=\frac{x^2}{2}+\frac{t^2}{2} \tag{23}$$

3.1.2. Least square method

Since trial function must satisfy the initial condition in Eq. (17), so we obtain the trial function as

$$u(t,x)=\frac{t^2}{2}+ax^2+bx^3+cx^4 \tag{24}$$

In this example (Eq. (16)), by the Eq. (8), residual function would be

$$R(t,x)=2a+6b+12cx^2-ax^2-bx^3-cx^4-1+0.5x^2 \tag{25}$$

Substitution the residual function (Eq. (25)) into Eq. (14), a system of three equations will be obtained and by solving this system of equations, coefficients *a*-*c* will be achieved

$$u(t,x)=\frac{t^2}{2}+0.49999x^2+2.5037\times10^{-9}x^3-1.2234\times10^{-9}x^4 \tag{26}$$

For this example, the results obtained by DTM and LSM are in good agreement with the exact solution as shown in Fig. 2.

3.2. Example2

Consider two- dimensional nonlinear wave equation with the initial condition are given by [27]

$$\frac{\partial^2 u}{\partial x^2}-u\frac{\partial^2 u}{\partial t^2}=-t^2\sin(t)-2t^2\sin^2(x), \qquad 0\le x,t\le 1, \tag{27}$$

$$u(t,0)=\frac{t^2}{2} \ , \qquad \frac{\partial}{\partial x}u(t,0)= \ 0 \ . \tag{28}$$

And the exact solution for this example is given by [27]

$$u(t,x)=t^2\sin x. \tag{29}$$

3.2.1. Differential transformation method

Taking the differential transform of Eq. (27), we have

$$(h+1)(h+2)U(k,h+2)=\sum_{s=0}^{h}\left(\sum_{r=0}^{k}U(r,h-s).(k-r+1)(k-r+2)U(k-r+2,s)\right)-$$
$$\sum_{s=0}^{h}\left(\sum_{r=0}^{k}\frac{\delta(s-0)\delta(k-r)\delta(r-2)\sin(\frac{(h-s)\pi}{2})}{(h-s)!}\right)$$
$$\sum_{s=0}^{h}\left(\sum_{r=0}^{k}\frac{2^{(h-s)}\delta(r-2)\delta(k-r)\delta(s-0)\cos(\frac{(h-s)\pi}{2}}{(h-s)!}\right)-\qquad(30)$$
$$\delta(k-2)\delta(h-0),\qquad k,h=0,1,2,\ldots,N$$

where $U(k,h)$ is the differential transform of $u(t,x)$.

Taking the differential transform from initial condition Eq. (28), and by recursive method, would be

$$U(i,0)=0\qquad,\ i=0,1,2,3,4,\ldots,m,\qquad(31)$$
$$U(2,1)=1\ ,\qquad U(i,1)=0\qquad,\ i=0,1,3,4,\ldots,n\qquad(32)$$

Substituting Eq. (31) and Eq. (32) into Eq. (30)the results are obtained as follow

$$U(2,3)=\frac{-1}{6}\qquad(33)$$
$$U(2,5)=\frac{1}{120}\qquad(34)$$
$$U(2,7)=\frac{-1}{5040}\qquad(35)$$
$$,$$
$$\vdots$$

Substituting all $U(k,h)$ into Eq. (5), we obtain the infinite series solution as follow

$$u(x,t)=t^2x-\frac{x^3t^2}{3!}+\frac{x^5t^2}{5!}-\frac{x^7t^2}{7!}+\ldots\qquad(36)$$

3.2.2. Least square method

Since trial function must satisfy the initial condition in Eq. (28), so we obtain the trial function as

$$u(t,x)=t^2x+at^2x^2+bt^2x^3+ct^2x^4\qquad(37)$$

In this example (Eq. (27)), by the Eq. (8), residual function would be

$$R(t,x)=-t^2(-2a-6bx-12cx^2+2x^2+4ax^3+4bx^4+4cx^5+2a^2x^4+4abx^5+$$
$$4acx^6+2b^2x^6+4bcx^7+2c^2x^8-\sin x-2\sin^2 x)\qquad(38)$$

Substitution the residual function (Eq. (38)) into Eq. (14), a system of three equations will be obtained and by solving this system of equations, coefficients *a*-*c* will be achieved

$$u(t,x)=t^2x+0.00376t^2x^2-0.18198t^2x^3+0.01961t^2x^4\qquad(39)$$

For this example, the results obtained by DTM and LSM are in good agreement with the exact solution as shown in Fig. 3. Also, Fig.4 depicts the results between the exact and DTM solution in difference number of *N*. It can be easily deduced from Fig.4 that higher terms of *N*are required to obtain more accuracy solution of DTM in comparison with numerical solution.

3.3. Example3

Finally, consider the coupled system are given by [27]

$$\frac{\partial^2 u}{\partial x^2} - v\frac{\partial^2 u}{\partial t^2} - u\frac{\partial^2 v}{\partial t^2} = 2 - 2x^2 - 2t^2 \tag{40}$$

$$\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 v}{\partial t^2} + u\frac{\partial^2 u}{\partial t^2} = 1 + \frac{3x^2}{2} + \frac{3t^2}{2} \tag{41}$$

With the initial conditions are

$$u(t,0) = t^2 \;, \quad \frac{\partial}{\partial x}u(t,0) = 0 \;, \quad v(t,0) = \frac{t^2}{2} \;, \quad \frac{\partial}{\partial x}v(t,0) = 0 \;. \tag{42}$$

And the exact solution for this example is given by [27]

$$\begin{aligned} u(t,x) &= x^2 + t^2, \\ v(t,x) &= \frac{x^2+t^2}{2}. \end{aligned} \tag{43}$$

3.3.1. Differential transformation method

Taking the differential transform of Eq. (40) and Eq. (41), we obtain as

$$\begin{aligned} (h+1)(h+2)U(k,h+2) = {} & \sum_{s=0}^{h}\left(\sum_{r=0}^{k} V(r,h-s).(k-r+1)(k-r+2)U(k-r+2,s)\right) + \\ & \sum_{s=0}^{h}\left(\sum_{r=0}^{k} U(r,h-s).(k-r+1)(k-r+2)V(k-r+2,s)\right) + \\ & 2\delta(h-0)\delta(k-0) - 2\delta(h-2).\delta(k-0) - 2\delta(k-2)\delta(h-0), \qquad k,h = 0,1,2,\ldots N \end{aligned} \tag{44}$$

$$\begin{aligned} (h+1)(h+2)V(k,h+2) = {} & \sum_{s=0}^{h}\left(\sum_{r=0}^{k} V(r,h-s).(k-r+1)(k-r+2)V(k-r+2,s)\right) - \\ & \sum_{s=0}^{h}\left(\sum_{r=0}^{k} U(r,h-s).(k-r+1)(k-r+2)V(k-r+2,s)\right) + \\ & \delta(h-0)\delta(k-0) + \frac{3}{2}\delta(h-2).\delta(k-0) + \frac{3}{2}\delta(k-2)\delta(h-0), \qquad k,h = 0,1,2,\ldots N \end{aligned} \tag{45}$$

where $U(k,h)$ and $V(k,h)$ are the differential transform of $u(t,x)$ and $v(\mathrm{t,x})$, respectively.

With transformed initial conditionsEq. (42),and by recursive method, would be

$$U(2,0) = 1 \quad , \quad U(i,0) = 0 \qquad , \qquad i = 0,1,3,\ldots,n, \tag{46}$$

$$U(i,1) = 0 \qquad , \qquad i = 0,1,2,\ldots,m, \tag{47}$$

$$V(2,0) = \frac{1}{2} \quad , \quad V(i,0) = 0 \quad , \quad i = 0,1,3,\ldots,n, \tag{48}$$

$$V(i,1) = 0 \qquad , \qquad i = 0,1,2,\ldots,m. \tag{49}$$

Substituting Eq. (46)-Eq. (49) into Eq. (44)and Eq. (45) we obtain the following

$$U(0,2) = 1 \quad (50)$$

$$V(0,2) = \frac{1}{2} \quad (51)$$

And the others of $U(k,h)$ and $V(k,h)$ are zero.

Substituting all $U(k,h)$ and $V(k,h)$ into Eq.(5), $u(x,t)$ and $v(x,t)$ can be determined. We have

$$u(x,t) = x^2 + t^2, \quad (52)$$

$$v(x,t) = \frac{x^2 + t^2}{2}. \quad (53)$$

3.3.2. Least square method

Since trial function must satisfy the initial condition in Eq. (42), so we obtain the trial function as

$$u(t,x) = t^2 + a_1 x^2 + a_2 x^3 + a_3 x^4 \quad (54)$$

$$v(t,x) = \frac{t^2}{2} + a_4 x^2 + a_5 x^3 + a_6 x^4 \quad (55)$$

For this example, we have two coupled equations (Eq.(40) and Eq.(41)), by the Eq. (8), two residual functions will be appeared as

$$R_1(a_1,a_2,a_3,a_4,a_5,a_6,t,x) = 2a_1 + 6a_2 x + 12a_3 x^2 - 2a_4 x^2 - 2a_5 x^3 - 2a_6 x^4 - a_1 x^2 - a_2 x^3 - a_3 x^4 - 2 + 2x^2 \quad (56)$$

$$R_2(a_1,a_2,a_3,a_4,a_5,a_6,t,x) = 2\mathrm{a}_4 + 6a_5 x + 12a_6 x^2 - \mathrm{a}_4 x^2 - a_5 x^3 - a_6 x^4 + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 - 1 - 1.5\mathrm{x}^2 \quad (57)$$

Substitution the residual function (Eq. (56) and Eq. (57)) into Eq. (14), a system of six equations will be obtained and by solving this system of equations, coefficients $a_1 - a_6$ will be achieved

$$u(t,x) = t^2 + x^2 - 1.289\times10^{-9} x^3 + 6.99\times10^{-10} x^4 \quad (58)$$

$$v(t,x) = 0.5t^2 + 0.5x^2 - 4.338\times10^{-9} x^3 + 2.06\times10^{-10} x^4 \quad (59)$$

For this example, the results obtained by DTM and LSM are in good agreement with the exact solution as shown in **Error! Reference source not found.**.

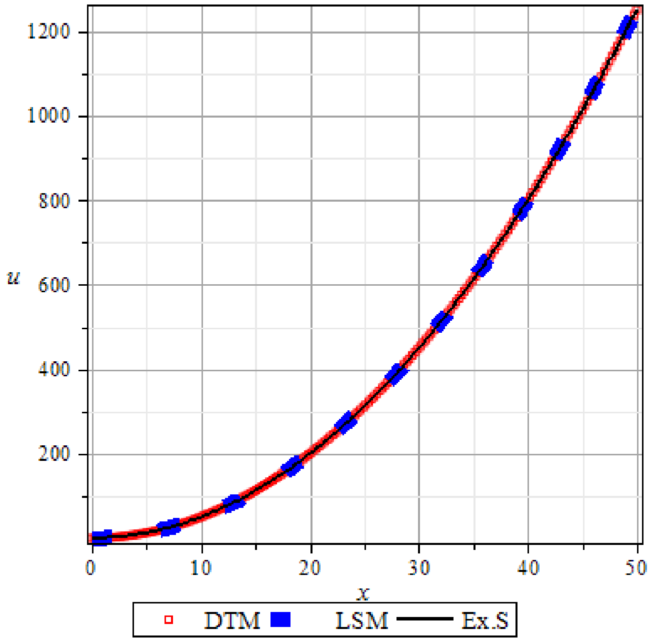

Fig. 2. Comparison of DTM, LSM and exact solution of example.1. Here t=1.

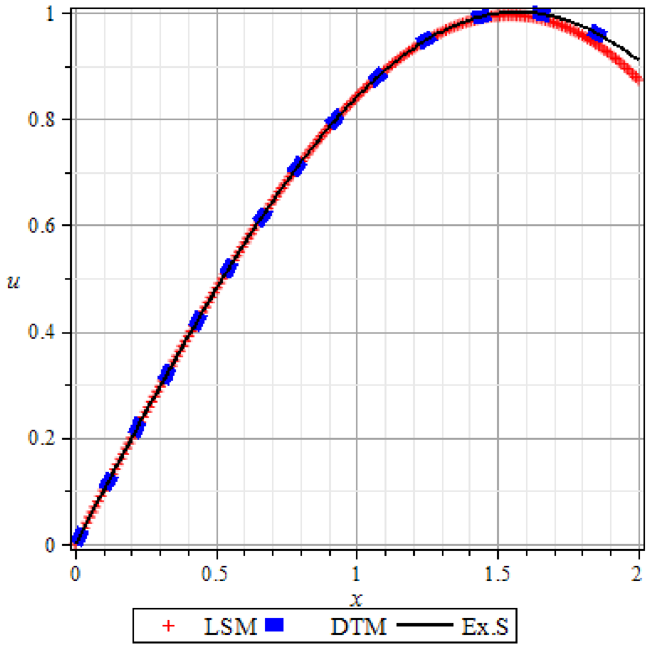

Fig. 3. Comparison of DTM, LSM and exact solution of example.2. Here t=1.

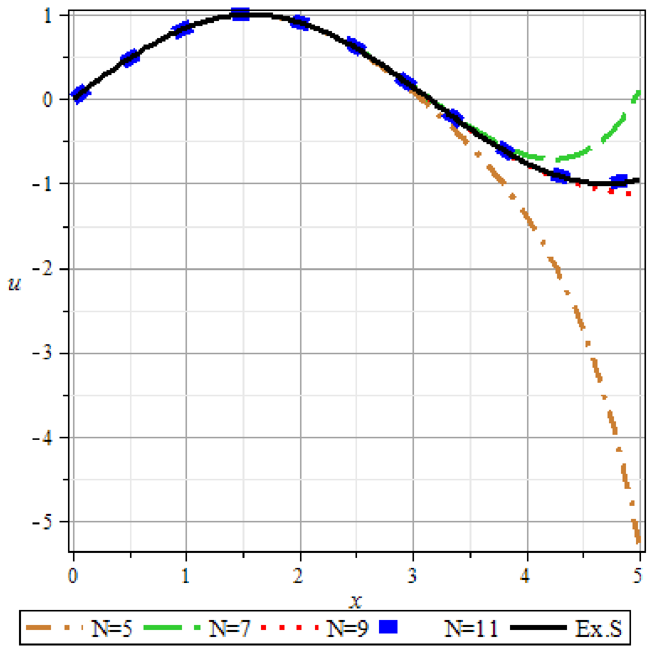

Fig.4. Comparison between Exact and DTM solution of example2, for different values of *N*. Here *t*=0.5.

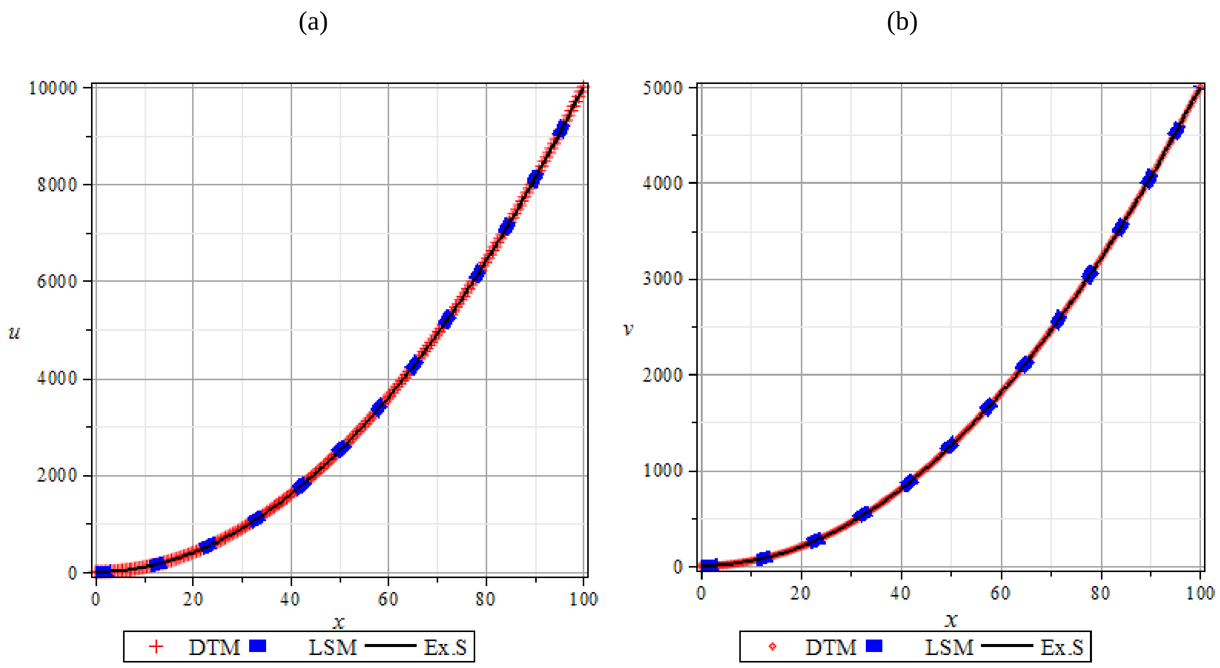

Fig. 5.Comparison of DTM, LSM and exact solution of example.3; a) *u (t, x)*, b) *v (t, x)*. Here *t*=0.5

4. Conclusion

In this article, we successfully applied two- dimensional transformation method (2D DTM) and least square method (LSM) for obtaining approximate solution of nonlinear differential wave equation. These methods have applied to three examples successfully, and the results achieved by these methods are excellent agreement with the exact solution of each example.We expect that these methods to two- dimensional nonlinear wave equations will be useful in solving other two- dimensional nonlinear equations.

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