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The Relationship Between Some Kinds of Ideal in The Order

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Abstract: This work will discuss one of the structures in Mathematics Algebra, namely Order. Simply put, order is a ring that certain criteria. For R is a ring which is of order, defining the R-ideal is difference with defining ideal (regular) in R as it is known in general. An R-ideal in R is certainly an ideal (regular) in R. However, in general, an ideal (regular) in R is not an R-ideal in R. However, in certain circumstances, the ideal (regular) in R is also an R-ideal. In addition to R-ideal, in order also known notion some other ideal. In this paper will be discussed the relationship between several types of ideal in the order.

Keywords: ideal, invertible, order, reflexive, quotient ring.

1. Introduction

This paper will discuss one of the structures in Mathematics Algebra, that is Order. Further, some kind of ideal that is closely associated with the order and types of linkages between these ideals will be the focus of study. Simply put, the order is a ring that meets certain criteria. For defining the order necessary for the understanding of the quotient ring and some other sense. Moreover, the definition of the quotient ring requires understanding of regular elements. Therefore, the order begins with understanding the definition of a regular element in a ring.

In the ring R which is an order known some ideal sense, such as R-ideal, fractional R-ideal, reflexive ideal, invertible ideal, and v-ideal. Furthermore, for R is a ring which is an order, the definition of R-ideal in R different from the defining ideal (regular) in R as it is known in general. An R-ideal in R is a ideal (regular) in R. However, in general, an ideal (regular) in R is not a R-ideal in R.

This paper will describe the notion of ideal types referred to in paragraph above. Apart from presenting the ideal type, is presented as well as some theory that links between the order and these ideals.

2. Definition, symbol, and Basic Theory

This study is a literature review of studies that use methods of adaptation and exploitation. Therefore, in this section are presented some sense, the basic theories, and the results of studies of several researchers who will adapted and exploited.

Definition. 2.1 [Zariski dan Samuel, 1958]

Let R be a ring. An element $0 \neq x \in R$ is called right regular if $xr = 0$ implies $r = 0$. While the left regular element is defined similar. If $x \in R$ is a right and left regular element, then $x \in R$ is called reguler.

The set of all regular elements in a ring form a set which is closed under multiplication and this set contains the identity element of R. The set is called multiplicative set. In general, a subset of a ring which is closed under multiplication, contains the identity element, and does not contain zero element is called multiplicative set.

Reguler elements in a ring does not necessarily have an inverse in the ring. This encourages the undefined quotient ring, which ring contains elements that revert all regular elements with specific propries.

Let Q be a ring that contains the ring R and the inverse of all regular elements in R. The ring Q is called the right quotient ring of R, if every $q \in Q$ can be written $q = rs^{-1}$ for an $r \in R$ and s is a regular element in R. The right quotient ring of R is defined similar. A ring Q arena called the quotient ring of R if Q is a right and left quotient ring of R.

Furthermore, the ring which is the quotient ring of the ring itself is called the quotient ring. Thus, it can be concluded that a ring Q is called the quotient ring, if every regular element is a unit element.

Observing the process of defining the quotient ring of a ring, it appears that not every ring has a quotient ring. Associated with the existence of the quotient ring, there are necessary and sufficient condition of a ring which has a quotient ring. Terms are granted by understanding the conditions Ore.

Let S be a subset of the ring R which is closed under multiplication. The set S is said to satisfy the right Ore condition if, for each $r \in R$ and $s \in S$ there exist $r_1 \in R$ and $s_1 \in S$ such that $rs_1 = sr_1$. Left Ore condition is defined similar. Furthermore, the ring R which satisfy the right (left) Ore condition for $S = R$ is called right (left) Ore ring.

Using the Ore condition above, the following necessary and sufficient conditions are presented ring that has a quotient ring.

Lemma 2.1 [McConnell and Robson, 1987]

1. A ring with identity element which does not contain divisor of zero element has a right quotient ring if and only if it is a right Ore domain.
2. A right Noetherian ring with identity element which does not contain divisor of zero element is a right Ore domain.

Using Lemma 2.1 we can conclude that the right Noetherian ring with identity element which is not contain divisor of zero elements has a quotient ring.

Furthermore, relooking at the quotient ring, it was found that two different ring may have the same quotient ring. For example, the ring $k[x]$ and $k[x, x^{-1}]$. This phenomenon inspired the definition of order.

Let Q be the quotient ring. Subring $R \subseteq Q$ is called the right order in Q if every $q \epsilon Q$ in the form $q = rs^{-1}$ for some $r, s \in R$. So also for the order left, subring $R \subseteq Q$ is called the left order in Q if every $q \epsilon Q$ in the form $q = s^{-1}r$ for some $r, s \in R$. If R is a right order once the left order, then R is called an order.

In the quotient ring, order is not unique. This encourages defines the maximum order.

Definition 2.2 [McConnell dan Robson, 1987]

Let Q be a quotient ring and $R_1, R_2 \subseteq Q$ are right orders in Q. Relation $\sim$ is defined with $R_1 \sim R_2$ if there exist $a_1, a_2, b_1, b_2 \in Q$ unit in Q such that $a_1R_1b_1 \subseteq R_2$ and $a_2R_2b_2 \subseteq R_1$.

It is clear that the relation ~ in Definition 2.2 is an equivalence relation. These relationships will form the equivalent classes. Order right order R is called right-maximal if R maximum in the equivalent class. Similar maximal left order defined. While R is called maximal order if R is a maximal order right and left.

Several types of order are defined in the order or closely related to the order presented in this section. Ideal types of order in question, among others, fractional ideal, invertible ideal, and v-ideal. Apart from presenting the ideal type, is presented also some theories that found links between the order and these ideals.

Definition 2.3

Let R be an order in the quotient ring Q. Right submodule I of Q_R that meet $aI \subseteq R$ and $bR \subseteq I$ for some unit $a, b \subseteq Q$ is called fractional right R-ideal. Fractional left R-ideal is defined similar. If I is a left and right R-ideal fractional, then I called a fractional R-ideal. Furthermore, if I is an R-fractional ideal right and $aI \subseteq R$, then I called right R-ideal. The same is true for left R-ideal. R-ideal that once left and right R-ideal is called R-ideal.

Using Definition of R-ideal above is not the same as the defining ideal (regular) in R as it is known in general. An ideal (regular) I in R is not necessarily a R-ideal in R, because the unit element $b \in Q$ that satisfy $bR \subseteq I$ do not necessarily exist. However, in certain circumstances, the ideal (regular) in R is also an R-ideal.

Here, some definitions and notations used in the theory of order. Suppose that R is order in the ring Q. For the sets of X and Y of Q, is defined (Marubayashi, Miyamoto, and Ueda, 1997),

$$(X, Y)_r = \{q \in Q \mid Yq \subseteq X\}$$

$$(X, Y)_l = \{q \in Q \mid qY \subseteq X\}$$

$$X^{-1} = \{q \in Q \mid XqX \subseteq X\}.$$

For right fractional R-ideal I of Q, denoted

$$O_r(I) = (I{:}\, I)_r = \{q \in Q \mid Iq \subseteq I\}.$$

For left fractional R-ideal I of Q, denoted

$$O_l(I) = (I{:}\, I)_l = \{q \in Q \mid qI \subseteq I\}.$$

They are called right order and left order of I respectively.

Using the above definitions and notation, the relationship between the maximum order, fractional ideal, and R-ideal is given in the following theorem.

Theorem 2.2 [McConnell dan Robson (1987)]

If $R$is a right order in Q then the following conditions are equvqlent:

a. R is a maximal right order
b. $O_r(I) = O_l(I) = R$ for all fractional R-ideal I.
c. $O_r(I) = O_l(I) = R$ for all R-ideal I.

Fractional ideal, as defined in Definition 2.3, was further developed into an invertible ideal and v-ideal.

Definition 2.4 [Marubayashi, Miyamoto, dan Ueda, 1997]

A fractional R-ideal I is called right v-ideal if $I_v = I$ where $I_v = (R{:}\,(R{:}\,I)_r)_l$. Similarly, fractional R-ideal J is called left v-ideal if ${}_vJ = J$ where ${}_vJ = (R{:}\,(R{:}\,I)_l)_r$. A fractional R-ideal I is called v-ideal, if $I_v = I = {}_vI$. Meanwhile, a fractional R-ideal I is called invertible if $(R{:}\,I)_l I = R = I(R{:}\,I)_r$.

Apart from the invertible ideal and v-ideal, fractional ideal can also be developed into a reflexive ideal. To define the following notation is required reflexive ideal. Suppose R is a right order in the quotient ring Q and I is a fractional right R-ideal, denoted

$$I^* = (R{:}\, I)_l = \{q \in Q \mid qI \subseteq R\}.$$

Apart from the notations, the following theorem is needed to clarify the definition of reflexive ideal.

Theorem 2.3 [McConnell and Robson, 1987]

If R and R' are maximal orders in quotient ring Q and I is a fractional R-ideal, then $(R:I)_l = (R':I)_r$.

Using Theorem 2.3 and the notation I^*, the reflexive ideal is expressed as follows.

Definition 2.4 [McConnell and Robson, 1987]

Let R be an order in the quotient ring Q and I be a fractional R-ideal. If $I = I^{**}$, then I is said reflexive.

Observing the sense of reflexive ideal and v-ideal, the relationship between them is obtained as shown the following lemma.

Lemma 2.4

R be an order in the quotient ring Q and I be a fractional R-ideal. Then I reflexive if and only if I is a v -ideal.

Proof:

Using Theorem 2.2, we obtain

$$(R:(R:I)_l)_l = I = (R:(R:I)_r)_r.$$

On the other hand, Theorem 2.3 stats that $(R:I)_l = (R:I)_r$. Therefore we obtain the following:

$(R:(R:I)_l)_l = I$ if and only if $(R:(R:I)_l)_r = I = (R:(R:I)_r)_l$.

This completes the proof. ■

Lemma 2.4 has presented the link between v-ideal with a reflexive ideal. In addition to the reflexive ideal, it turns out, v-ideal is also associated with invertible ideal. To prove the links between them, the following lemma is required.

Lemma 2.5 [Marubayashi, Miyamoto, dan Ueda, 1997]

If I is an invertible ideal, then $(R:I)_l = I^{-1} = (R:I)_r$.

Furthermore, the linkage between the invertible ideal with v-ideal is given in the following lemma and it can be proved using Lemma 2.5.

Lemma 2.6

Let R be a ring with identity element and I is an ideal in R. If I is an invertible ideal, then I is a v-ideal.

Proof:

Let I be an invertible ideal, then $(R:I)_l = I^{-1} = (R:I)_r$. For $q \in I$, $(R:I)_l q = I^{-1}q \subseteq R$. This means $q \in I_v = (R:R:I)_l)_r$. So $I \subseteq I_v$. Conversely, let $q \in I_v$, then $q \in I$. So $I_v \subseteq I$. Therefore we get $I_v = I$. with similar way, we can show that ${}_vI = I$. This implies ${}_vI = I = I_v$ or I is a v-ideal. ■

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