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The rate of χ-space defined by a modulus

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Abstract: In this paper we introduce the modulus function of χ_π. We establish some inclusion relations, topological results and we characterize the duals of the χ_f^π sequence spaces.

1. Introduction

A complex sequence, whose kth term is x_k is denoted by $\{x_k\}$ or simply x. Let w be the set of all sequences $x=(x_k)$ and φ be the set of all finite sequences. Let l_∞, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x=(x_k)$ respectively. In respect of l_∞, c, c_0 we have $\|x\| = \sup_k |x_k|$, where $x=(x_k)\in c_0 \subset c \subset l_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $\lim_{k\to\infty} |x_k|^{\frac{1}{k}}$. The vector space of all entire sequences will be denoted by Γ. χ was discussed in Kamthan [5]. Matrix transformation involving χ were characterized by Sridhar [14] and Sirajiudeen [13]. Let χ_f^π be the set of all those sequences $x=(x_k)$ such that $\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}} \to 0$ as $k\to\infty$. Then χ_f^π is a metric space with the metric

$$d(x,y) = \sup_k \left\{ \left(k!\left|\frac{x_k - y_k}{\pi_k}\right|\right)^{\frac{1}{k}} ; k = 1,2,3,\ldots \right\}$$

Orlicz [11] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (1 \le p < \infty)$. Subsequently the different classes of sequence spaces were defined by Parashar and Choudhary [4], Mursaleen et al. [9], Bektas and Altin [1], Tripathy et al. [15], Rao and Subramanian [3] and many others.

The Orlicz sequence spaces is the special case of Orlicz space, studied in Ref [6].

Recall [6, 11] an Orlicz function is a function $M: [0,\infty] \to [0,\infty]$ which is continuous, non-decreasing and convex with $M(0)=0, M(x)>0$, for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. If the convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$ then this function is called modulus function, introduced by Nakano [10] and further discussed by Ruckle [12] and Maddox [8] and many others.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $k>0$, such that $M(2u) \le KM(u)\ (u \ge 0)$. The Δ_2-condition is equivalent to $M(lu) \le klM(u)$, for all values of u and for $l > 1$ Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$l_M = \left\{x \in w: \sum_{k=1}^{\infty} M\left(\left|\frac{x_k}{\pi_k}\right|\right) < \infty \text{ for some } \pi_k > 0\right\} \tag{1}$$

The space l_M with the norm

$$\|x\| = \inf\left\{\pi_k > 0: \sum_{k=1}^{\infty} M\left(\left|\frac{x_k}{\pi_k}\right|\right) \le 1\right\} \tag{2}$$

becomes a Banach space which is called an Orlicz sequence space. For$M(t) = t^p, 1 \le p < \infty$, the space l_M coincide with the classical sequence space l_p. Given a sequence $x = \{x_k\}$ its nth section is the sequence $x^{(n)} = \{x_1, x_1, \dots, x_n, 0,0, \dots\}, \delta^n = \left(0,0, \dots, \frac{\pi_k}{k!}, 0,0, \dots\right), \pi_k$ in the nth place and zero's elsewhere and $S^n = \left(0,0, \dots, \frac{\pi_k}{k!}, \frac{-\pi_k}{k!}, 0,0, \dots\right), \frac{\pi_k}{k!}$ in the nth place, $\frac{-\pi_k}{k!}$ in the $(n+1)$st place and zero's elsewhere. An FK-space (Frechet Coordinate Space) is a Frechet Space which is made up of numerical sequences and has the property that the coordinate functionals $P_k(x) = x_k (k = 1,2,3, \dots)$ are continuous. We recall the following definitions (see [16]).

An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric space (x, d) is said to have AK (or sectional convergence) if and only if $d\left(x^{(n)}, x\right) \to 0$ as $n \to \infty$ (see [16]). The space is said to have AD (or) be an AD space if φ is dense in X. We note that AK implies AD by [2].

If X is a sequence space, we define

1. $X' =$ the continuous dual of X;
2. $X^\alpha = \{a = (a_k): \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{for each } x \in X\}$;
3. $X^\beta = \{a = (a_k): \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in X\}$;
4. $X^\gamma = \left\{a = (a_k): \sup_n \left|\sum_{k=1}^{n} a_k x_k\right| < \infty, \text{for each } x \in X\right\}$;
5. Let be an FK-space $\supset \varphi$. Then $X^f = \left\{f\left(\delta^{(n)}\right): f \in X'\right\}$.

$X^\alpha, X^\beta, X^\gamma$ are called the α-(or Köthe Töeplitz) dual of X, β -(or generalized Köthe Tö eplitz) dual of X, γ-dual of X. Note that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$ then $Y^\mu \subset X^\mu$, for $\mu = \alpha, \beta$ or γ.

Lemma 1.1. (See[16, Theorem 7.27]). Let X be an FK space $\supset \varphi$. Then (i) $X^\gamma \subset X^f$. (ii) If X has AK, $X^\beta = X^F$. (iii) If X has A.D., $X^\beta = X^\gamma$.

2. Definition and Preliminaries

Let w denote the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$ and $f: [0, \infty) \to [0, \infty)$ be a modulus function.
Let

$$\chi_f^\pi = \left\{x \in w: \lim_{k \to \infty}\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)\right)^{\frac{1}{k}} = 0 \text{ for some } \pi_k > 0\right\}$$

$$\Gamma_f^\pi = \left\{x \in w: \lim_{k \to \infty}\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)\right)^{\frac{1}{k}} = 0 \text{ for some } \pi_k > 0\right\}$$

and

$$\Lambda_f^\pi = \left\{x \in w: \sup_k\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)\right)^{\frac{1}{k}} < \infty \text{ for some } \pi_k > 0\right\}$$

The space χ_f^π is a metric space with the metric

$$d(x, y) = \inf\left\{\pi_k > 0: \sup_k\left(f\left(k!\left|\frac{x_k - y_k}{\pi_k}\right|\right)\right)^{\frac{1}{k}} \le 1\right\} \tag{3}$$

The space Γ_f and Λ_f is a metric space with the metric

$$d(x,y) = \inf\left\{\pi_k > 0: \sup_k \left(f\left(\left|\frac{x_k - y_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq 1\right\} \tag{4}$$

3. Main Result

Proposition 3.1.

$\chi_f^\pi \subset \Gamma_f^\pi$ with the hypothesis that $f\left(\left|\frac{x_k}{\pi_k}\right|^{\frac{1}{k}}\right) \leq f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}$.

Proof. Let $x \in \chi_f^\pi$. Then we have the following implications

$$f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \to 0 \text{ as } k \to \infty. \tag{5}$$

But $f\left(\left|\frac{x_k}{\pi_k}\right|^{\frac{1}{k}}\right) \leq f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)$; by our assumption, implies that

$\Longrightarrow f\left(\left|\frac{x_k}{\pi_k}\right|^{\frac{1}{k}}\right) \to 0$ as $k \to \infty$ by (5)

$\Longrightarrow x \in \Gamma_f^\pi$

$\Longrightarrow \chi_f^\pi \subset \Gamma_f^\pi$.

This completes the proof.

Proposition 3.2.

χ_f^π has AK where f is a modulus function.

Proof. Let $x = \{x_k\} \in \chi_f^\pi$, then $\left\{f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right\} \in \chi_f^\pi$ and hence

$$\sup_{k \geq n+1} f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \to 0 \text{ as } n \to \infty \tag{6}$$

$d\left(x, x^{[n]}\right) = \sup_{k \geq n+1} f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \to 0$ as $n \to \infty$ by using (6).

$\Longrightarrow x^{[n]} \to x$ as $n \to \infty$,

implying that χ_f^π has AK. This completes the proof.

Proposition 3.3.

χ_f^π is solid.

Proof. Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \chi_f^\pi$. $f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)$, because f is non-decreasing. But $f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \in \chi$, because $y \in \chi_f^\pi$. That is, $f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \to 0$ as $k \to \infty$ and $f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \to 0$ as $k \to \infty$.

Therefore, $x = \{x_k\} \in \chi_f^\pi$. This completes the proof.

Proposition 3.4.

Let f be a modulus function which satisfies Δ_2-condition. Then $\chi \subset \chi_f^\pi$.

Proof. Let

$$x \in \chi \tag{7}$$

Then $\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq \epsilon$ sufficiently large k and every $\epsilon > 0$. By taking $\pi_k \geq \frac{1}{2}$. $f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq f\left(\frac{\epsilon}{\pi_k}\right) \leq f(2\epsilon)$ (because f is non-decreasing)

$$f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq kf(\epsilon) \tag{8}$$

by Δ_2-condition, for some $k \geq 0 \leq \epsilon$. $f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \to 0$ as $k \to \infty$ (by defining $f(\epsilon) < \frac{\epsilon}{k}$). Hence $x \in \chi_f^{\pi}$. From (7) and since

$$x \in \chi_f^{\pi}, \tag{9}$$

we get $\chi \subset \chi_f^{\pi}$. This completes the proof.

Proposition 3.5.

If f is a modulus function, then χ_f^{π} is linear space over the set of complex number C.

Proof. Let $x, y \in \chi_f^{\pi}$ and $\alpha, \beta \in C$. In order to prove the result we need to find some π_k such that

$$f\left(\left(k!\left|\frac{\alpha x_k + \beta y_k}{\pi_k}\right|\right)\right)^{\frac{1}{k}} \to 0 \text{ as } k \to \infty \tag{10}$$

Since, $x, y \in \chi_f^{\pi}$ such that

$$f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \to 0 \text{ as } k \to \infty \tag{11}$$

Since f is a non-decreasing modulus function, we have

$$f\left(\left(k!\left|\frac{\alpha x_k + \beta y_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq f\left(\left(k!\left|\frac{\alpha x_k}{\pi_k}\right|\right)^{\frac{1}{k}} + \left(k!\left|\frac{\beta y_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq f\left(|\alpha|\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}} + |\beta|\left(k!\left|\frac{y_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)$$

Take π_k such that $\frac{1}{\pi_k} = \min\left\{\frac{1}{|\alpha|}\frac{1}{\pi_1}, \frac{1}{|\beta|}\frac{1}{\pi_2}\right\}$. Then

$$f\left(\left(k!\left|\frac{\alpha x_k + \beta y_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq f\left(\left(k!\left|\frac{x_k}{\pi_1}\right|\right)^{\frac{1}{k}} + \left(k!\left|\frac{y_k}{\pi_2}\right|\right)^{\frac{1}{k}}\right) \to 0 \text{ by (11).}$$

Hence $f\left(k!\left|\frac{\alpha x_k + \beta y_k}{\pi_k}\right|\right)^{\frac{1}{k}} \to 0$ as $k \to \infty$. So $(\alpha x + \beta y) \in \chi_f^{\pi}$. Therefore, χ_f^{π} is linear. This completes the proof.

Definition 3.6.

Let $P = (P_k)$ be any sequence of positive real numbers. Then we define $\chi_f^{\pi}(P) = \left\{x = (x_k): f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}} \to 0 \text{ as } k \to \infty\right\}$. Suppose that P_k is a constant for all k, the $\chi_f^{\pi}(P) = \chi_f^{\pi}$.

Proposition 3.7.

Let $0 \leq p_k \leq q_k$ and let $\left\{\frac{q_k}{p_k}\right\}$ be bounded. Then $\chi_f^{\pi}(q) = \chi_f^{\pi} t(p)$.

Proof. Let

$$x \in \chi_f^{\pi}(q), \tag{12}$$

$$\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k} \to 0 \text{ as } k \to \infty \tag{13}$$

Let $t_k = \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$.

Since $p_k \leq q_k$, we have $0 \leq \lambda_k \leq 1$.

Take $0 < \lambda < \lambda_k$. Define

$$u_k = \begin{cases} t_k, & (t_k \geq 1) \\ 0, & (t_k < 1) \end{cases} \text{ and } v_k = \begin{cases} 0, & (t_k \geq 1) \\ t_k, & (t_k < 1) \end{cases} \tag{14}$$

$t_k = u_k + v_k; {t_k}^{\lambda_k} = {u_k}^{\lambda_k} + {v_k}^{\lambda_k}$. Now it follows that ${u_k}^{\lambda_k} \leq u_k \leq t_k$ and ${v_k}^{\lambda_k} \leq {v_k}^{\lambda}$. Since ${t_k}^{\lambda_k} = {u_k}^{\lambda_k} + {v_k}^{\lambda_k}$, then ${t_k}^{\lambda_k} \leq t_k + {v_k}^{\lambda}$.

$$\left(f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k}\right)^{\lambda_k} \leq \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k}$$

$$\Rightarrow \left(f\left(\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k}\right)^{p_k/q_k} \leq \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k}$$

$$\Rightarrow \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \leq \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k}$$

But $\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k} \to 0$ as $k \to \infty$ by (13)

Therefore $\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \to 0$ as $k \to \infty$. Hence

$$x \in \chi_f^{\pi}(p) \tag{15}$$

From (12) and (15) we get $\chi_f^{\pi}(q) \subset \chi_f^{\pi}(p)$.
Thus completes the proof.

Proposition 3.8.

(a) Let $0 \leq \inf_{p_k} \leq p_k \leq 1$. Then $\chi_f^{\pi}(p) \subset \chi_f^{\pi}$.

(b) Let $1 \leq p_k \leq \sup_{p_k} < \infty$. Then $\chi_f^{\pi} \subset \chi_f^{\pi}(p)$.

Proof.

(a) Let $x \in \chi_f^{\pi}(p)$

$$\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \to 0 \text{ as } k \to \infty \tag{16}$$

Since $0 \leq \inf_{p_k} \leq p_k \leq 1$.

$$\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \leq \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \tag{17}$$

From (16) and (17) it follows that $x \in \chi_f^{\pi}$. Thus $\chi_f^{\pi}(p) \subset \chi_f^{\pi}$. We have thus proven (a).

(b) Let $p_k \geq 1$ for each k and $\sup_{p_k} < \infty$.

Let $x \in \chi_f^{\pi}$

$$\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \to 0 \text{ as } k \to \infty \tag{18}$$

Since $1 \leq p_k \leq \sup_{p_k} < \infty$ we have

$$\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \leq \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) \tag{19}$$

$\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \to 0$ as $k \to \infty$ by using (18). Therefore $x \in \chi_f^\pi(p)$. This completes the proof.

Proposition 3.9.

Let $0 \le p_k \le q_k < \infty$ for each k. Then $\chi_f^\pi(p) \subseteq \chi_f^\pi(q)$.

Proof. Let $x \in \chi_f^\pi(p)$.

$$\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \to 0 \text{ as } k \to \infty \tag{20}$$

This implies that $\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \le 1$ for sufficiently large k.

Since f is non-decreasing, we get

$$\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k} \le \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{p_k} \tag{21}$$

$$\Rightarrow \left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^{q_k} \to 0 \text{ as } k \to \infty \ (\text{by using } (20))$$

$$x \in \chi_f^\pi(q)$$

Hence, $\chi_f^\pi(p) \subseteq \chi_f^\pi t(q)$.

This completes the proof.

Proposition 3.10.

$\chi_f^\pi(p)$ is a r-convex for all r where $0 \le r \le \inf_{p_k}$. Moreover if $p_k = p \le 1\ \forall k$, then they are p-convex.

Proof. We shall prove the proposition for $\chi_f^\pi(p)$. Let $x \in \chi_f^\pi(p)$ and $r \in \left(0, \lim_{n\to\infty} p_n\right)$. Then, there exists k_0 such that $r \le p_k, \forall k > k_0$. Now, define

$$g^*(x) = \inf\left\{\pi_k : f\left(\left(k!\left|\frac{x_k - y_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^r + f\left(\left(k!\left|\frac{x_k - y_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right)^r\right\}, \tag{22}$$

since, $r \le p_k \le 1, \forall k > k_0$. g^* is subadditive. Further, for $0 \le |\lambda| \le 1$; $|\lambda|^{p_k} \le |\lambda|^r, \forall k > k_0$.

$$g^*(\lambda x) \le |\lambda|^r g^*(x) \tag{23}$$

Now, for $0 < \delta < 1$,

$$U = \{x : g^*(x) \le \delta\}, \text{which is an absolutely } r - \text{convex set} \tag{24}$$

for

$$|\lambda|^r + |\mu|^r \le 1; x, y \in U \tag{25}$$

Now,

$$g^*(\lambda x + \mu y) \le g^*(\lambda x) + g^*(\mu y) \le |\lambda|^r g^*(x) + |\mu|^r g^*(y) \le |\lambda|^r \delta + |\mu|^r \delta \text{ using (23) and (24)}$$
$$\le (|\lambda|^r + |\mu|^r)\delta \le 1.\delta, \text{by using (25)} \le \delta$$

If $p_k = p \le 1\ \forall k$ then for $0 < r < 1$, $U = \{x : g^*(x) \le \delta\}$ is an absolutely p-convex set. This can be obtained by a similar analysis and therefore we omit the details. This completes the proof.

Proposition 3.11.

$$\left(\chi_f^\pi\right)^\beta = \Lambda_f^\pi$$

Proof.

Step 1: $\chi_f^\pi \subset \Gamma_f^\pi$ by Proposition 3.1;

$\Rightarrow \left(\Gamma_f^\pi\right)^\beta \subset \left(\chi_f^\pi\right)^\beta$. But $\left(\Gamma_f^\pi\right)^\beta = \Lambda_f^\pi$ see (3).

$$\Lambda_f^\pi \subset \left(\chi_f^\pi\right)^\beta \tag{26}$$

Step 2: Let $y \in \left(\chi_f\right)^\beta$ we have $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \chi_f^\pi$. We recall that $S^{(k)}$ has $\frac{1}{k!}$ in the kth place and zero's elsewhere, with $x = S^{(k)}$, $\left(f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}}\right) = \left\{0,0,\dots,f\left(\frac{(1)^{\frac{1}{k}}}{\pi_k}\right),0,\dots\right\}$ which converges to zero. Hence, $S^{(k)} \in \chi_f^\pi$. Hence, $d\left(S^{(k)},0\right) = 1$. But $|y_k| \le \|f\| d\left(S^{(k)},0\right) < \infty\ \forall k$. Thus (y_k) is a bounded rate sequence and hence a rate analytic sequence.
In other words $y \in \Lambda_f^\pi$.

$$\left(\chi_f^\pi\right)^\beta \subset \Lambda_f^\pi \tag{27}$$

Step 3: From (25) and (26) we obtain $\left(\chi_f^\pi\right)^\beta = \Lambda_f^\pi$. This completes the proof.
Proposition 3.12.
$\left(\chi_f^\pi\right)^\mu = \Lambda$ for $\mu = \alpha,\beta,\gamma,f$.
Proof.
Step 1: χ_f has AK by Proposition 3.2. Hence, by Lemma 1.1 (ii).
We get $\left(\chi_f^\pi\right)^\beta = \left(\chi_f^\pi\right)^f$. But $\left(\chi_f^\pi\right)^\beta = \Lambda_f^\pi$.
Hence

$$\left(\chi_f^\pi\right)^f = \Lambda_f^\pi \tag{28}$$

Step 2: Since AK⟹AD. Hence by Lemma 1.1.(iii).
We get $\left(\chi_f^\pi\right)^\beta = \left(\chi_f^\pi\right)^\gamma$. Therefore

$$\left(\chi_f^\pi\right)^\gamma = \Lambda_f^\pi \tag{29}$$

Step 3: χ_f^π is normal by Proposition 3.3. Hence by Proposition ?? and (12), we get

$$\left(\chi_f^\pi\right)^\alpha = \left(\chi_f^\pi\right)^\gamma = \Lambda_f^\pi \ (30) \tag{30}$$

From (28) and (30) we have $\left(\chi_f^\pi\right)^\alpha = \left(\chi_f^\pi\right)^\beta = \left(\chi_f^\pi\right)^\gamma = \left(\chi_f^\pi\right)^f = \Lambda_f^\pi$.
Proposition 3.13.
The dual space of χ_f^π is Λ. In other words $\chi_f^* = \Lambda$.
Proof.
We recall that $S^{(k)}$ has $\frac{\pi_k}{k!}$ in the kth place and zero's elsewhere with

$$x = S^{(k)}, \qquad f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}} = \left\{0,0,\dots,f\left(\frac{(1)^{\frac{1}{k}}}{\pi_k}\right),0,\dots\right\}$$

Hence, $S^{(k)} \in \chi_f^\pi$. We have $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \chi_f^\pi$ and $f \in \left(\chi_f^\pi\right)^\alpha$where χ_f^π is the dual space of χ_f^π. Take $x = S^{(k)} \in \chi_f^\pi$. Then

$$|y_k| \le \|f\| d\left(S^{(k)},0\right) < \infty \text{ for all } k. \tag{31}$$

Thus (y_k) is a bounded rate sequence and hence a rate of analytic sequence. In other words, $y \in \Lambda$. Therefore $\chi_f^* = \Lambda$. This completes the proof.
Lemma 3.14 ([16, Theorem 8.6.1]).
$Y \supset X \Leftrightarrow Y^f \subset X^f$ where X is an AD-space and Y on FK-space.
Proposition 3.15.
Let Y be any FK-space $\supset \varphi$. Then $Y \supset \chi_f^\pi$ if and only if the sequence $S^{(k)}$ is weakly analytic.
Proof. The following implications establish the result

$Y \supset \chi_f^\pi \Leftrightarrow Y^f \subset \chi_f^\pi$ since χ_f has AD by Lemma 3.14
$\Leftrightarrow$ for each $f \in Y'$, the topological dual of Y.
$\Leftrightarrow f\left(S^{(k)}\right)$ is rate of analytic.
$\Leftrightarrow S^{(k)}$ is weakly rate of analytic.
This completes the proof.

Proposition 3.16.

χ_f^π is a complete metric space under the metric

$$d(x,y) = \sup_k \left\{ f\left(k!\left|\frac{x_k - y_k}{\pi_k}\right|\right)^{\frac{1}{k}} : k = 1,2,3,\ldots \right\}$$

Where $x = (x_k) \in \chi_f^\pi$ and $y = (y_k)\chi_f^\pi$.

Proof. Let $\{x^{(n)}\}$ be Cauchy sequence in χ_f^π. Then given any $\epsilon > 0$ there exists a positive integer N depending on ϵ such that $d\left(x^{(n)}, x^{(m)}\right) < \epsilon$ for all $n \geq N$ and for $m \geq N$. Hence, $\sup_k \left\{ f\left(k!\left|\frac{x_k^{(n)} - x_k^{(m)}}{\pi_k}\right|\right)^{\frac{1}{k}} \right\} < \epsilon$ for all $n \geq N$ and for $m \geq N$.

Consequently $f\left(k!\left|\frac{x_k^{(n)}}{\pi_k}\right|\right)^{\frac{1}{k}}$ is a Cauchy sequence in the metric space C of a complex numbers.
But C is complete. So,

$$f\left(k!\left|\frac{x_k^{(n)}}{\pi_k}\right|\right)^{\frac{1}{k}} \to f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}} \text{ as } n \to \infty.$$

Hence there exists a positive integer no such that

$$\sup_k \left\{ f\left(k!\left|\frac{x_k^{(n)} - x_k}{\pi_k}\right|\right)^{\frac{1}{k}} \right\} < \epsilon \text{ for all } n \leq n_0.$$

In particular, we have

$$f\left(k!\left|\frac{x_k^{(n)} - x_k}{\pi_k}\right|\right)^{\frac{1}{k}} < \epsilon.$$

Now

$$f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}} \leq f\left(k!\left|\frac{x_k - x_k^{(n_0)}}{\pi_k}\right|\right) + f\left(k!\left|\frac{x_k^{(n_0)}}{\pi_k}\right|\right)^{\frac{1}{k}} < \epsilon \to 0 \text{ as } k \to \infty.$$

Thus

$$f\left(k!\left|\frac{x_k}{\pi_k}\right|\right)^{\frac{1}{k}} < \epsilon \to 0 \text{ as } k \to \infty.$$

That is $x \in \chi_f^\pi$.
Therefore χ_f^π is a complete metric space. This completes the proof.

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