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Traveling Wave Solutions of the RLW and Boussinesq Equations

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Abstract: In this study, we use the generalized tanh function method for the traveling wave solutions of the generalized regularized long-wave (gRLW) equation and Boussinesq equation system.

Key Words: The generalized regularized long-wave equation, Boussinesq equation system, generalized tanh function method, traveling wave solutions.

1 Introduction

The mathematical modeling of events in nature can be explained by differential equations. These equations are mathematical models of complex physical occurrences that arise in engineering, chemistry, biology, mechanics and physics. So, the theory of nonlinear dispersive wave motion has recently undergone much study. The solutions of nonlinear equations play a crucial role in applied mathematics and physics, because; solutions of nonlinear partial differential equations provide a very significant contribution to people about the nature of physical phenomenon. We do not attempt to characterize the general form of nonlinear dispersive wave equations [1, 2]. Furthermore, when an original nonlinear equation is directly calculated, the solution will preserve the actual physical characters of solutions [3]. Explicit solutions to the nonlinear equations are of fundamental importance. Various methods for obtaining explicit solutions to nonlinear evolution equations have been proposed. Many explicit exact methods have been introduced in literature [4-17]. Among them are Generalized Miura Transformation, Darboux Transformation, Cole-Hopf Transformation, Hirota's dependent variable Transformation, the inverse scattering Transform and the Backlund Transformation, sine-cosine method, Painleve method, homogeneous balance method, and similarity reduction method.

Traveling wave solutions of many nonlinear differential equations can be stated with tanh function terms [18, 19]. The tanh function terms firstly were used on base *ad hoc* in 1990 and 1991 [20, 21]. Then, Malfliet [22] formalized the tanh method in 1992 and illustrated it with several examples, Parkes and Duffy presented the automatic tanh method [23] in 1996, after, Fan defined the extended tanh method [24] in 2000, later Elwakil presented the modified extended tanh method [25] in 2002, separately, the generalized extended tanh method [26] by Zheng in 2003, the improved extended tanh method [27] by Yomba in 2004, the tanh function method [28] by Chen and Zhang in 2004.

In this study, we implement the generalized tanh function method [29] to obtain the traveling wave solutions of the gRLW equation

$$u_t + u_x + \gamma(u^p)_x - \beta u_{xxt} = 0 \quad (1)$$

where p is a positive integer and γ and β are positive constants. The Eq. (1) was first put forward as a model for small-amplitude long-waves on the surface of water in a channel by peregrine [30, 31] and later by Benjamin et al [32]. In physical situations such as unidirectional wave's propagation in a water channel, long-crested waves in near-shore zones, and many others, the RLW equation serves an alternative model to the KdV equation [33, 34].

Furthermore, we study traveling wave solutions of general Boussinesq (gBQ) type fluid model. There are many types of equation form of the Boussinesq equation system, one of them is given by

$$u_t + v_x + uu_x + pu_{xxt} = 0$$

$$v_t + (uv)_x + \beta u_{xxx} = 0 \quad (2)$$

(2) where p and β are real constant. This system which generalizes the classical Boussinesq equation system was derived by Sachs [35] to describe small amplitude long waves in a water channel.

2 An Analysis of the Method and applications

In this chapter, we give a simple description of the tanh function method. For doing this, it can be considered in a two variables general form of nonlinear PDE

$$Q(u, u_t, u_x, u_{xx}, \dots) = 0 \quad (3)$$

The solution of the equation (3) is expressed as a finite series of tanh functions

$$u(x,t) = \sum_{i=0}^{M} a_i(x,t) F^i(\xi) \quad (4)$$

where $\xi = \xi(x,t) = \alpha x + q(t)$, M is a positive integer that can be determined by balancing the highest order derivate and with the highest nonlinear terms in equation, $a_0(x,t), a_1(x,t), \dots, a_n(x,t)$ and $\xi(x,t)$ can be determined. Substituting solution (4) into Eq. (3) yields a set of algebraic equations for F^i, then, all coefficients of F^i have to vanish. After this separated algebraic equation, we could found coefficients $a_0(x,t), a_1(x,t), \dots, a_n(x,t)$.

In this work, we consider to solve the Boussinesq equation and gRLW by using the generalized tanh function method which is introduced by Chen and Zhang [29]. The fundamental of their method is to take full advantage of the Riccati equation that tanh function satisfies and use its solutions. The required Riccati equation is given as

$$F' = A + BF + CF^2 \quad (5)$$

where $F' = \frac{dF}{d\xi}$ and A, B, C are constants. Some of the solutions are given the paper [29].

3 Applications of the Method

Example 1. Let's consider a gRLW equation,

$$u_t + u_x + \gamma(u^p)_x - \beta u_{xxt} = 0 \quad (6)$$

Let $\gamma = 1, \beta = 1$ and $p = 2$, we have equation

$$u_t + u_x + 2uu_x - u_{xxt} = 0 \quad (7)$$

When balancing uu_x with u_{xxt} then gives $M = 2$. Therefore, we may choose

$$u = f(t) + g(t)F(\xi) + h(t)F^2(\xi) \quad (8)$$

Where $\xi = \alpha x + q(t)$. Substituting (8) into Eq. (7) yields a set of algebraic equations for $f(t), g(t), h(t)$ and A, B, C. These systems are finding as

$$f_t + gq_tA + gA\alpha + 2fgA\alpha - g_t\alpha^2AB - g\alpha^2q_tAB^2 - 2\alpha^2A^2Cgq_t - 2h_t\alpha^2A^2 - 6h\alpha^2A^2Bq_t = 0$$

$$g_t + gq_tB + 2hAq_t + gB\alpha + 2hA\alpha + 2fgB\alpha + 4fhA\alpha + 2g^2A\alpha - g_t\alpha^2B^2 - g\alpha^2q_tB^3 -$$

$$-8\alpha^2ABCgq_t - 2g_t\alpha^2AC - 6h_t\alpha^2AB - 14h\alpha^2AB^2q_t - 16h\alpha^2A^2Cq_t = 0$$

$$gq_tC + h_t + 2hBq_t + gC\alpha + 2hB\alpha + 2fgC\alpha + 4fhB\alpha + 2g^2B\alpha + 6hgA\alpha - 3g_t\alpha^2BC - 7g\alpha^2q_tB^2C - 8\alpha^2AC^2gq_t - 8h_t\alpha^2AC - 52h\alpha^2ABCq_t - 8h\alpha^2B^3q_t - 4h_t\alpha^2B^2 = 0$$

$$2hCq_t + 2hC\alpha + 4fhC\alpha + 2g^2C\alpha + 6hgB\alpha + 4h^2A\alpha - 2\alpha^2C^2g_t - 12g\alpha^2q_tBC^2 - 38h\alpha^2B^2Cq_t - 40h\alpha^2AC^2q_t - 10h_t\alpha^2BC = 0 \quad (9)$$

$$6hgC\alpha + 4h^2B\alpha - 6\alpha^2C^3gq_t - 54h\alpha^2BC^2q_t - 6h_t\alpha^2C^2 = 0$$

$$4h^2C\alpha - 24h\alpha^2C^3q_t = 0$$

From the solutions of the system, we can found

$$h = 6\alpha C^2q_t,\ \ g = 6BC\alpha q_t,\ \ f = \frac{8\alpha^2ACq_t + \alpha^2B^2q_t - q_t - \alpha}{2\alpha},\ \ q_{tt} = 0 \quad (10)$$

with the aid of Mathematica. From (10), we can get

$$q = \lambda t,\ \ q_t = \lambda,\ \ h = 6\alpha C^2\lambda,\ \ g = 6BC\alpha\lambda,\ \ f = \frac{8\alpha^2AC\lambda + \alpha^2B^2\lambda - \lambda - \alpha}{2\alpha},\ \ \lambda = const. \quad (11)$$

Substituting (10) and (11) into (8) we have obtained the following multiple soliton-like and triangular periodic solutions (including rational solutions) of equation (7). These solutions are:

Case 1: *if* $A = C = 1$,

$$u = \frac{8\alpha^2\lambda - \lambda - \alpha}{2\alpha} + 6\alpha\lambda tan^2(\xi) \quad (12)$$

Case 2: *if* $A = C = -1$,

$$u = \frac{8\alpha^2\lambda - \lambda - \alpha}{2\alpha} + 6\alpha\lambda cot^2(\xi) \quad (13)$$

Case 3: *if* $A = 1, C = -1$,

$$\begin{cases} u = \frac{-8\alpha^2\lambda - \lambda - \alpha}{2\alpha} + 6\alpha\lambda tanh^2(\xi) \\ u = \frac{-8\alpha^2\lambda - \lambda - \alpha}{2\alpha} + 6\alpha\lambda coth^2(\xi) \end{cases} \quad (14)$$

Case 4: *if* $A = C = -\frac{1}{2}$,

$$\begin{cases} u = \frac{2\alpha^2\lambda - \lambda - \alpha}{2\alpha} + \frac{3\alpha\lambda}{2}\left(cot(\xi) \pm csc(\xi)\right)^2 \\ u = \frac{2\alpha^2\lambda - \lambda - \alpha}{2\alpha} + \frac{3\alpha\lambda}{2}\left(sec(\xi) - tan(\xi)\right)^2 \\ u = \frac{2\alpha^2\lambda - \lambda - \alpha}{2\alpha} + \frac{3\alpha\lambda cot^2(\xi)}{2\left(1 \pm csc(\xi)\right)^2} \end{cases} \quad (15)$$

Case 5: *if* $A = C = \frac{1}{2}$,

$$\begin{cases} u = \frac{2\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{3\alpha\lambda}{2}\left(tan(\xi) \pm sec(\xi)\right)^2 \\ u = \frac{2\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{3\alpha\lambda}{2}\left(csc(\xi) - cot(\xi)\right)^2 \\ u = \frac{2\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{3\alpha\lambda tan^2(\xi)}{2\left(1\pm sec(\xi)\right)^2} \end{cases} \tag{16}$$

Case 6: *if* $A = \frac{1}{2}, C = -\frac{1}{2}$

$$\begin{cases} u = \frac{-2\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{3\alpha\lambda}{2}\left(coth(\xi) \pm csch(\xi)\right)^2 \\ u = \frac{-2\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{3\alpha\lambda}{2}\left(tanh(\xi) \pm isech(\xi)\right)^2 \\ u = \frac{-2\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{3\alpha\lambda tanh^2(\xi)}{2\left(1\pm sech(\xi)\right)^2} \\ u = \frac{-2\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{3\alpha\lambda coth^2(\xi)}{2\left(1\pm icsch(\xi)\right)^2}, \quad i^2 = -1 \end{cases}$$

(17)

Case 7: *if* $A = 1,\ B = -2,\ C = 2$

$$u = \frac{20\alpha^2\lambda-\lambda-\alpha}{2\alpha} - \frac{24\alpha\lambda tan(\xi)}{\left(1+tan(\xi)\right)} + \frac{24\alpha\lambda tan^2(\xi)}{\left(1+tan(\xi)\right)^2} \tag{18}$$

Case 8: *if* $A = 1,\ B = C = 2,$

$$u = \frac{20\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{24\alpha\lambda tan(\xi)}{\left(1-tan(\xi)\right)} + \frac{24\alpha\lambda tan^2(\xi)}{\left(1-tan(\xi)\right)^2} \tag{19}$$

Case 9: *if* $A = -1,\ B = 2,\ C = -2$

$$u = \frac{20\alpha^2\lambda-\lambda-\alpha}{2\alpha} - \frac{24\alpha\lambda cot(\xi)}{\left(1+cot(\xi)\right)} + \frac{24\alpha\lambda cot^2(\xi)}{\left(1+cot(\xi)\right)^2} \tag{20}$$

Case 10: *if* $A = -1,\ B = C = -2$

$$u = \frac{20\alpha^2\lambda-\lambda-\alpha}{2\alpha} + \frac{24\alpha\lambda cot(\xi)}{\left(1-cot(\xi)\right)} + \frac{24\alpha\lambda cot^2(\xi)}{\left(1-cot(\xi)\right)^2} \tag{21}$$

Case 11: *if* $A = B = 0$ and $C \neq 0,$

$$u = \frac{-\alpha-\lambda}{2\alpha} + 6\alpha\lambda C^2\left(-\frac{1}{C\xi+c_0}\right)^2 \tag{22}$$

where $\xi = \alpha x + \lambda t$ for (12) - (22).

Example 2. Let's consider Boussinesq equation system,

$$\begin{cases} u_t + v_x + uu_x + pu_{xxt} = 0 \\ v_t + (uv)_x + \beta u_{xxx} = 0 \end{cases} \tag{23}$$

Let $p = 1, \beta = 1$, we have system of the equation

$$\begin{cases} u_t + v_x + uu_x + u_{xxt} = 0 \\ v_t + u_x v + v_x u + u_{xxx} = 0 \end{cases} \tag{24}$$

When balancing uu_x with u_{xxt} then gives $M_1 = 2$ and $u_x v$, $v_x u$ with u_{xxx} then gives $M_2 = 2$. Therefore, we may choose

$$\begin{cases} u = f(t) + g(t)F(\xi) + h(t)F^2(\xi) \\ v = f_1(t) + g_1(t)F(\xi) + h_1(t)F^2(\xi) \end{cases} \tag{25}$$

where $\xi = \alpha x + q(t)$. Substituting (25) into Eq. (24) yields a set of algebraic equations for $f(t), g(t), h(t), f_1(t), g_1(t), h_1(t)$. These systems are finding as

$$f_t + gq_tA + g_1A\alpha + fg\alpha + g_t\alpha^2AB + g\alpha^2q_tAB^2 + 2\alpha^2A^2Cgq_t + 2h_t\alpha^2A^2 + 6h\alpha^2A^2Bq_t = 0$$

$$g_t + gq_tB + 2hAq_t + g_1B\alpha + 2h_1A\alpha + fgB\alpha + 2fhA\alpha + g^2A\alpha + g_t\alpha^2B^2 + g\alpha^2q_tB^3 + 8\alpha^2gABCq_t + 2\alpha^2ACg_t + 6h_t\alpha^2AB + 14h\alpha^2AB^2q_t + 16h\alpha^2A^2Cq_t = 0$$

$$gq_tC + h_t + 2hBq_t + g_1C\alpha + 2h_1B\alpha + fgC\alpha + 2fhB\alpha + g^2B\alpha + 3hgA\alpha + 7g\alpha^2B^2Cq_t + 3\alpha^2BCg_t + 8\alpha^2AC^2gq_t + 52h\alpha^2ABCq_t + 8h_t\alpha^2AC + 4h_t\alpha^2B^2 + 8h\alpha^2B^3q_t = 0$$

$$2hCq_t + 2h_1C\alpha + 2fhC\alpha + g^2C\alpha + 3hgB\alpha + 2h^2A\alpha + 12\alpha^2gBC^2q_t + 2\alpha^2C^2g_t + 40h\alpha^2AC^2q_t + 38h\alpha^2B^2Cq_t + 10h_t\alpha^2BC = 0$$

$$3hgC\alpha + 2h^2B\alpha + 6\alpha^2C^3gq_t + 54h\alpha^2BC^2q_t + 6h_t\alpha^2C^2 = 0$$

$$2h^2C\alpha + 24h\alpha^2C^3q_t = 0$$

$$(f_1)_t + g_1Aq_t + gf_1A\alpha + g_1fA\alpha + g\alpha^3AB^2 + 2\alpha^3A^2Cg + 6h\alpha^3A^2B = 0$$

$$(g_1)_t + g_1Bq_t + 2h_1Aq_t + gf_1B\alpha + 2hf_1A\alpha + 2gg_1A\alpha + g_1fB\alpha + 2h_1fA\alpha + g\alpha^3B^3 + 8\alpha^3gABC + 14h\alpha^3AB^2 + 16h\alpha^3A^2C = 0$$

$$g_1Cq_t + (h_1)_t + 2h_1Bq_t + gf_1C\alpha + 2hf_1B\alpha + 2gg_1B\alpha + 3hg_1A\alpha + 3gh_1A\alpha + g_1f_1C\alpha + 2h_1fB\alpha + 7g\alpha^3B^2C + 8\alpha^3AC^2g + 52h\alpha^3ABC + 8h\alpha^3B^3 = 0$$

$$2h_1Cq_t + 2hf_1C\alpha + 2gg_1C\alpha + 3hg_1B\alpha + 3gh_1B\alpha + 4hh_1A\alpha + 2h_1fC\alpha + 12\alpha^3gBC^2 + 40h\alpha^3AC^2 + 38h\alpha^3B^2C = 0$$

$$3hg_1C\alpha + 3gh_1C\alpha + 4hh_1B\alpha + 6\alpha^3C^3g + 54h\alpha^3BC^2 = 0$$

$$4hh_1C\alpha + 24h\alpha^3C^3 = 0$$

(26)

From the solutions of the system, we can found

$$h = -12\alpha C^2q_t,\ \ g = -12BC\alpha q_t,\ \ f = -\frac{q_t}{\alpha} - \frac{\alpha}{2q_t} - B^2\alpha q_t - 8AC\alpha q_t,\ \ q_{tt} = 0$$

$$h_1 = -6\alpha^2C^2,\ \ g_1 = -6BC\alpha^2,\ \ f_1 = -\frac{\alpha^2B^2}{2} - 4\alpha^2AC + \frac{\alpha^2}{4(q_t)^2} \tag{27}$$

with the aid of Mathematica. From (27), we can get

$$q = \lambda t,\ \ q_t = \lambda,\ \ h = -12\alpha C^2\lambda,\ \ g = -12BC\alpha\lambda$$

$$f = -\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - B^2\alpha\lambda - 8\alpha\lambda AC,\ \ f_1 = -\frac{\alpha^2B^2}{2} - 4\alpha^2AC + \frac{\alpha^2}{4\lambda^2} \tag{28}$$

Substituting (27) and (28) into (25) we have obtained the following multiple soliton-like and triangular periodic solutions (including rational solutions) of equation (24) same as the above example. These solutions are:

Case 1: *if* $A = C = 1$,

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - 8\alpha\lambda\right) - 12\alpha\lambda tan^2(\xi) \\ v = \left(-4\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - 6\alpha^2 tan^2(\xi) \end{cases} \quad (29)$$

Case 2: *if* $A = C = -1$,

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - 8\alpha\lambda\right) - 12\alpha\lambda cot^2(\xi) \\ v = \left(-4\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - 6\alpha^2 cot^2(\xi) \end{cases} \quad (30)$$

Case 3: *if* $A = 1,\ C = -1$,

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} + 8\alpha\lambda\right) - 12\alpha\lambda tanh^2(\xi) \\ v = \left(4\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - 6\alpha^2 tanh^2(\xi) \end{cases} \quad \begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} + 8\alpha\lambda\right) - 12\alpha\lambda coth^2(\xi) \\ v = \left(4\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - 6\alpha^2 coth^2(\xi) \end{cases} \quad (31)$$

Case 4: *if* $A = C = -\frac{1}{2}$,

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - 2\alpha\lambda\right) - 3\alpha\lambda\left(cot(\xi) \pm csc(\xi)\right)^2 \\ v = \left(-\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\alpha^2\left(cot(\xi) \pm csc(\xi)\right)^2 \end{cases}$$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - 2\alpha\lambda\right) - 3\alpha\lambda\left(sec(\xi) - tan(\xi)\right)^2 \\ v = \left(-\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\alpha^2\left(sec(\xi) - tan(\xi)\right)^2 \end{cases} \quad (32)$$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - 2\alpha\lambda\right) - \frac{3\alpha\lambda cot^2(\xi)}{\left(1 \pm csc(\xi)\right)^2} \\ v = \left(-\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\frac{\alpha^2 cot^2(\xi)}{\left(1 \pm csc(\xi)\right)^2} \end{cases}$$

Case 5: *if* $A = C = \frac{1}{2}$,

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - 2\alpha\lambda\right) - 3\alpha\lambda\left(tan(\xi) \pm sec(\xi)\right)^2 \\ v = \left(-\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\alpha^2\left(tan(\xi) \pm sec(\xi)\right)^2 \end{cases}$$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - 2\alpha\lambda\right) - 3\alpha\lambda\left(csc(\xi) - cot(\xi)\right)^2 \\ v = \left(-\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\alpha^2\left(csc(\xi) - cot(\xi)\right)^2 \end{cases} \quad (33)$$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} - 2\alpha\lambda\right) - \frac{3\alpha\lambda tan^2(\xi)}{\left(1 \pm sec(\xi)\right)^2} \\ v = \left(-\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\frac{\alpha^2 tan^2(\xi)}{\left(1 \pm sec(\xi)\right)^2} \end{cases}$$

Case 6: *if* $A = \frac{1}{2},\ C = -\frac{1}{2}$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} + 2\alpha\lambda\right) - 3\alpha\lambda\left(coth(\xi) \pm csch(\xi)\right)^2 \\ v = \left(\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\alpha^2\left(coth(\xi) \pm csch(\xi)\right)^2 \end{cases}$$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} + 2\alpha\lambda\right) - 3\alpha\lambda\left(tanh(\xi) \pm isech(\xi)\right)^2 \\ v = \left(\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\alpha^2\left(tanh(\xi) \pm isech(\xi)\right)^2 \end{cases} \tag{34}$$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} + 2\alpha\lambda\right) - \frac{3\alpha\lambda tanh^2(\xi)}{\left(1 \pm sech(\xi)\right)^2} \\ v = \left(\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\frac{\alpha^2 tanh^2(\xi)}{\left(1 \pm sech(\xi)\right)^2} \end{cases}$$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda} + 2\alpha\lambda\right) - \frac{3\alpha\lambda coth^2(\xi)}{\left(1 \pm icsch(\xi)\right)^2} \\ v = \left(\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{3}{2}\frac{\alpha^2 coth^2(\xi)}{\left(1 \pm icsch(\xi)\right)^2} \end{cases}$$

Case 7: *if* $A = 1,\ B = -2,\ C = 2$

$$\begin{cases} u = \left(-20\alpha\lambda - \frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda}\right) + \frac{48\alpha\lambda tan(\xi)}{1 + tan(\xi)} - \frac{48\alpha\lambda tan^2(\xi)}{\left(1 + tan(\xi)\right)^2} \\ v = \left(-10\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) + \frac{24\alpha^2 tan(\xi)}{1 + tan(\xi)} - \frac{24\alpha^2 tan^2(\xi)}{\left(1 + tan(\xi)\right)^2} \end{cases} \tag{35}$$

Case 8: *if* $A = 1,\ B = C = 2$

$$\begin{cases} u = \left(-20\alpha\lambda - \frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda}\right) - \frac{48\alpha\lambda tan(\xi)}{1 - tan(\xi)} - \frac{48\alpha\lambda tan^2(\xi)}{\left(1 - tan(\xi)\right)^2} \\ v = \left(-10\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{24\alpha^2 tan(\xi)}{1 - tan(\xi)} - \frac{24\alpha^2 tan^2(\xi)}{\left(1 - tan(\xi)\right)^2} \end{cases} \tag{36}$$

Case 9: *if* $A = -1,\ B = 2,\ C = -2$

$$\begin{cases} u = \left(-20\alpha\lambda - \frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda}\right) + \frac{48\alpha\lambda cot(\xi)}{1 + cot(\xi)} - \frac{48\alpha\lambda cot^2(\xi)}{\left(1 + cot(\xi)\right)^2} \\ v = \left(-10\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) + \frac{24\alpha^2 cot(\xi)}{1 + cot(\xi)} - \frac{24\alpha^2 cot^2(\xi)}{\left(1 + cot(\xi)\right)^2} \end{cases} \tag{37}$$

Case 10: *if* $A = -1,\ B = -2,\ C = -2$

$$\begin{cases} u = \left(-20\alpha\lambda - \frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda}\right) - \frac{48\alpha\lambda cot(\xi)}{1 - cot(\xi)} - \frac{48\alpha\lambda cot^2(\xi)}{\left(1 - cot(\xi)\right)^2} \\ v = \left(-10\alpha^2 + \frac{\alpha^2}{4\lambda^2}\right) - \frac{24\alpha^2 cot(\xi)}{1 - cot(\xi)} - \frac{24\alpha^2 cot^2(\xi)}{\left(1 - cot(\xi)\right)^2} \end{cases} \tag{38}$$

Case 11: *if* $A = B = 0,\ C \neq 0$

$$\begin{cases} u = \left(-\frac{\lambda}{\alpha} - \frac{\alpha}{2\lambda}\right) - 12\alpha\lambda C^2\left(\frac{-1}{C\xi + c_0}\right)^2 \\ v = \frac{\alpha^2}{4\lambda^2} - 6\alpha^2 C^2\left(\frac{-1}{C\xi + c_0}\right)^2 \end{cases} \tag{39}$$

where $\xi = \alpha x + \lambda t$ for (29) - (39).

3 Conclusions

In this paper, we present the generalized tanh function method by using ansatz (4) and, with aid of Mathematica, implement it in a computer algebraic system. An implementation of the method is given by applying it to gRLW equation and Boussinesq equation system with physics interests. We also obtain some new and more general solutions for gRLW equation and Boussinesq equation system at same time. The method can be used to many other nonlinear equations or coupled ones. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer.

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