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Upper bound of the second Hankel determinant for a subclass of analytic functions

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Abstract: In the present investigation an upper bound of second Hankel determinant $|a_2a_4-a_3^2|$ for the functions belonging to the class $S_s^*(\alpha;A,B)$ is studied.

Keywords: Analytic functions, Subordination, Schwarz function, Second Hankel determinant.

1 Introduction

Let *A be* the class of analytic functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

in the unit disc $E=\{z:|z|<1\}$.

By *S*, we denote the class of functions $f(z)\in A$ and univalent in *E*.

U denotes the class of Schwarzian functions

$$w(z)=\sum_{k=1}^{\infty}p_kz^k$$

which are analytic in the unit disc $E=\{z:|z|<1\}$ and satisfying the conditions $w(0)=0$ and $|w(z)|<1$.

For two functions f and g which are analytic in *E*, f is said to be subordinate to *g* (symbolically $f\prec g$) if there exists a Schwarz function $w(z)\in U$, such that $f(z)=g(w(z))$.

$S_s^*(\alpha;A,B)$ denote the subclass of functions $f(z)\in A$ and satisfying the condition

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)}+\alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'}\prec\frac{1+Az}{1+Bz},-1\le B<A\le 1,0\le\alpha\le 1,z\in E. \tag{1.2}$$

The following observations are obvious:

(i) $S_s^*(\alpha;1,-1)\equiv S_s^*(\alpha)$, the class of $\alpha-$starlike functions with respect to symmetric points.

(ii) $S_s^*(0;1,-1) \equiv S_s^*$, the class of starlike functions with respect to symmetric points introduced by Sakaguchi [11].

(iii) $S_s^*(1;1,-1) \equiv K_s$, the class of convex functions with respect to symmetric points introduced by Das and Singh [1].

(iv) $S_s^*(0;A,B) \equiv S_s^*(A,B)$, the subclass of starlike functions with respect to symmetric points introduced and studied by Goel and Mehrok [2].

(v) $S_s^*(1;A,B) \equiv K_s(A,B)$, the subclass of convex functions with respect to symmetric points.

In 1976, Noonan and Thomas [9] stated the *qth* Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as second Hankel determinant:

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left| a_2 a_4 - a_3^2 \right|,$$

and obtain an upper bound to the functional $H_2(2)$ for $f(z) \in S_s^*(\alpha; A, B)$. Earlier Janteng et al.([3],[4],[5]), Mehrok and Singh [8], Singh ([12],[13]) and many others have obtained sharp upper bounds of $H_2(2)$ for different classes of analytic functions.

2 Preliminary Results

Let *P* be the family of all functions *p* analytic in *E for* which $\operatorname{Re}(p(z)) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

for $z \in E$.

Lemma 2.1. [10] If $p \in P$, then $|p_k| \leq 2 (k = 1, 2, 3, \ldots)$.

Lemma 2.2. [6,7] If $p \in P$, then

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right)x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z,$$

for some *x* and *z* satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

3 Main Result

Theorem 3.1. If $f \in S_s^*(\alpha; A, B)$, then

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{4(1+2\alpha)^2}. \tag{3.1}$$

Proof. If $f(z)\in S_s^*(\alpha;A,B)$, then there exists a Schwarz function $w(z)\in U$ such that

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)}+\alpha\frac{2\left(zf'(z)\right)'}{\left(f(z)-f(-z)\right)'}=\varphi\left(w(z)\right), \tag{3.2}$$

where

$$\varphi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-B(A-B)z^2+B^2(A-B)z^3+...$$

$$=1+B_1z+B_2z^2+B_3z^3+... \tag{3.3}$$

Define the function $p_1(z)$ by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+c_3z^3+... \tag{3.4}$$

Since $w(z)$ is a Schwarz function, we see that $\mathrm{Re}\left(p_1(z)\right)>0$ and $p_1(0)=1.$ Define the function $h(z)$ by

$$h(z)=(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)}+\alpha\frac{2\left(zf'(z)\right)'}{\left(f(z)-f(-z)\right)'}=1+b_1z+b_2z^2+b_3z^3+... \tag{3.5}$$

In view of the equations (3.2), (3.4) and (3.5), we have

$$h(z)=\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=\varphi\left(\frac{c_1z+c_2z^2+c_3z^3+...}{2+c_1z+c_2z^2+c_3z^3+...}\right)$$

$$=\varphi\left(\frac{1}{2}c_1z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)z^3+...\right)$$

$$=1+\frac{B_1c_1}{2}z+\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\right]z^2+\left[\frac{B_1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)+\frac{B_2c_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_3c_1^3}{8}\right]z^3+...$$

Thus,

$$b_1=\frac{B_1c_1}{2};b_2=\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\text{ and }b_3=\frac{B_1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)+\frac{B_2c_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_3c_1^3}{8}. \tag{3.6}$$

Using (3.3) and (3.5) in (3.6), we obtain

$$\left.\begin{aligned} a_2 &= \frac{(A-B)c_1}{4(1+\alpha)}, \\ a_3 &= \frac{(A-B)}{8(1+2\alpha)}\left[2c_2-(B+1)c_1^2\right], \\ a_4 &= \frac{(A-B)}{64(1+\alpha)(1+2\alpha)(1+3\alpha)}\left[\begin{array}{l} 8(1+\alpha)(1+2\alpha)c_3 \\ +2\left\{(1+5\alpha)A-\left[(1+5\alpha)+4(1+\alpha)(1+2\alpha)\right]B-4(1+\alpha)(1+2\alpha)\right\}c_1c_2 \\ +(B+1)\left\{\left[2(1+\alpha)(1+2\alpha)+(1+5\alpha)\right]B-(1+5\alpha)A+2(1+\alpha)(1+2\alpha)\right\}c_1^3 \end{array}\right] \end{aligned}\right\}. \quad (3.7)$$

(3.7) yields,

$$a_2a_4-a_3^2=\frac{(A-B)^2}{C(\alpha)}\left\{2Lc_1(4c_3)+Mc_1^2(2c_2)-Nc_1^4-4R\left(4c_2^2\right)\right\} \quad (3.8)$$

where $C(\alpha)=256(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)$,

$$L=(1+\alpha)(1+2\alpha)^2,$$

$$M=(1+2\alpha)(1+5\alpha)A+\left[8(1+\alpha)^2(1+3\alpha)-(1+2\alpha)(1+5\alpha)-4(1+\alpha)(1+2\alpha)^2\right]B$$
$$+\left[8(1+\alpha)^2(1+3\alpha)-4(1+\alpha)(1+2\alpha)^2\right],$$

$$N=(B+1)\left\{\begin{array}{l}(1+2\alpha)(1+5\alpha)A+\left[4(1+\alpha)^2(1+3\alpha)-2(1+\alpha)(1+2\alpha)^2-(1+2\alpha)(1+5\alpha)\right]B \\ +\left[4(1+\alpha)^2(1+3\alpha)-2(1+\alpha)(1+2\alpha)^2\right]\end{array}\right\}$$

and

$$R=(1+3\alpha)(1+\alpha)^2.$$

Using Lemma 2.1 and Lemma 2.2 in (3.8), we obtain

$$\left|a_2a_4-a_3^2\right|=\frac{(A-B)^2}{C(\alpha)}\left|\begin{array}{l} -\left\{(1+2\alpha)(1+5\alpha)AB+\left[4(1+\alpha)^2(1+3\alpha)-2(1+\alpha)(1+2\alpha)^2-(1+2\alpha)(1+5\alpha)\right]B^2\right\}c_1^4 \\ +\left\{(1+2\alpha)(1+5\alpha)A+\left[8(1+\alpha)^2(1+3\alpha)-(1+2\alpha)(1+5\alpha)-4(1+\alpha)(1+2\alpha)^2\right]B\right\}c_1^2\left(4-c_1^2\right)x \\ -2\left\{8(1+\alpha)^2(1+3\alpha)-\left[2(1+\alpha)^2(1+3\alpha)-(1+\alpha)(1+2\alpha)^2\right]c_1^2\right\}\left(4-c_1^2\right)x^2 \\ +4(1+\alpha)(1+2\alpha)^2c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z \end{array}\right|$$

Assume that $c_1=c$ and $c\in[0,2]$, using triangular inequality and $|z|\le 1$, we have

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{C(\alpha)}\left\{\begin{array}{l}\left\{2\left(4-c^2\right)\left[8(1+\alpha)^2(1+3\alpha)-\left(2(1+\alpha)^2(1+3\alpha)-(1+\alpha)(1+2\alpha)^2\right)c^2\right]-4(1+\alpha)(1+2\alpha)^2c\left(4-c^2\right)\right\}\delta^2\\+\left|(1+2\alpha)(1+5\alpha)A+\left[8(1+\alpha)^2(1+3\alpha)-(1+2\alpha)(1+5\alpha)-4(1+\alpha)(1+2\alpha)^2\right]B\right|\left(4-c^2\right)c^2\delta\\+\left|(1+2\alpha)(1+5\alpha)AB+\left[4(1+\alpha)^2(1+3\alpha)-2(1+\alpha)(1+2\alpha)^2-(1+2\alpha)(1+5\alpha)\right]B^2\right|c^4\\+4(1+\alpha)(1+2\alpha)^2c\left(4-c^2\right)\end{array}\right\}$$

$=\dfrac{(A-B)^2}{C(\alpha)}F(\delta)$, where $\delta=|x|\le1$ and

$$\begin{aligned}F(\delta)=&\left\{2\left(4-c^2\right)\left[8(1+\alpha)^2(1+3\alpha)-\left(2(1+\alpha)^2(1+3\alpha)-(1+\alpha)(1+2\alpha)^2\right)c^2\right]-4(1+\alpha)(1+2\alpha)^2c\left(4-c^2\right)\right\}\delta^2\\&+\left|(1+2\alpha)(1+5\alpha)A+\left[8(1+\alpha)^2(1+3\alpha)-(1+2\alpha)(1+5\alpha)-4(1+\alpha)(1+2\alpha)^2\right]B\right|\left(4-c^2\right)c^2\delta\\&+\left|(1+2\alpha)(1+5\alpha)AB+\left[4(1+\alpha)^2(1+3\alpha)-2(1+\alpha)(1+2\alpha)^2-(1+2\alpha)(1+5\alpha)\right]B^2\right|c^4\\&+4(1+\alpha)(1+2\alpha)^2c\left(4-c^2\right)\end{aligned}$$

is an increasing function. Therefore $Max.F(\delta)=F(1)$.

Consequently

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{C(\alpha)}G(c), \tag{3.9}$$

where

$$G(c)=F(1).$$

So $G(c)=S(\alpha)c^4+T(\alpha)c^2+64(1+\alpha)^2(1+3\alpha)$

where

$$S(\alpha)=\left\{\begin{array}{l}\left|(1+2\alpha)(1+5\alpha)AB+\left[4(1+\alpha)^2(1+3\alpha)-2(1+\alpha)(1+2\alpha)^2-(1+2\alpha)(1+5\alpha)\right]B^2\right|\\-\left|(1+2\alpha)(1+5\alpha)A+\left[8(1+\alpha)^2(1+3\alpha)-(1+2\alpha)(1+5\alpha)-4(1+\alpha)(1+2\alpha)^2\right]B\right|\\+2\left[2(1+\alpha)^2(1+3\alpha)-(1+\alpha)(1+2\alpha)^2\right]\end{array}\right\}$$

and

$$T(\alpha)=\left\{\begin{array}{l}4\left|(1+2\alpha)(1+5\alpha)A+\left[8(1+\alpha)^2(1+3\alpha)-(1+2\alpha)(1+5\alpha)-4(1+\alpha)(1+2\alpha)^2\right]B\right|\\-8\left[4(1+\alpha)^2(1+3\alpha)-(1+\alpha)(1+2\alpha)^2\right]\end{array}\right\}.$$

Now $G'(c)=4S(\alpha)c^3+2T(\alpha)c$ and $G''(c)=12S(\alpha)c^2+2T(\alpha)$.

$G'(c)=0$ gives

$$c\left[2S(\alpha)c^2+T(\alpha)\right]=0.$$

$G''(c)$ is negative at $c=0$.

So $Max.G(c)=G(1)$.

Hence from (3.9), we obtain (3.1).

The result is sharp for $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$.

For $A = 1$ and $B = -1$ in Theorem 3.1, we obtain the following result:

Corollary 3.1.1. If $f(z) \in S_s^*(\alpha)$, then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{(1+2\alpha)^2}.$$

For $\alpha = 0$, $A = 1$ and $B = -1$, Theorem 3.1 gives the following result due to Janteng et al.[5].

Corollary 3.1.2. If $f(z) \in S_s^*$, then

$$\left|a_2 a_4 - a_3^2\right| \le 1.$$

For $\alpha = 1$, $A = 1$ and $B = -1$, Theorem 3.1 gives the following result due to Janteng et al.[5].

Corollary 3.1.3. If $f(z) \in K_s$, then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{9}.$$

Putting $\alpha = 0$ in Theorem 3.1, we obtain the following result:

Corollary 3.1.4. If $f(z) \in S_s^*(A, B)$, then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{(A-B)^2}{4}.$$

Putting $\alpha = 1$ in Theorem 3.1, we obtain the following result:

Corollary 3.1.5. If $f(z) \in K_s(A, B)$, then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{(A-B)^2}{36}.$$

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