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Results on fuzzy soft functions

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Abstract: The concept of fuzzy soft function is mentioned by Aygünoğlu *et al* and Kharal *et al* in their papers (named Introduction to Fuzzy Soft Groups (2009)and Mappings on Fuzzy Soft Classes (2009), respectively). In this paper, some results on the fuzzy soft image and preimage of set theoretic operations of fuzzy soft sets under a fuzzy soft function are studied. Also the notion of fuzzy soft equality is introduced and some related results are given.

Keywords: Soft Set, Fuzzy Soft Set, Fuzzy Soft Functions, Fuzzy Soft Image and Preimage

1 Introduction

Fuzzy set theory was firstly proposed by researcher L.A. Zadeh [1] and has become a very important tool to solve problems which contains vagueness. It has been studied by both mathematicians and computer scientists over the years.

Soft set theory, which is a completely new approach for modeling uncertainty, was introduced by Molodtsov [2] in 1999. He established the fundamental results of this theory. Maji et al [3], Pei et al [4], Feng et al [5], Chen et al [6] Aktas et al [7] improved the work of Molodtsov [2].

Both fuzzy set theory and soft set theory deal with the problems which contain vagueness and uncertainties, from the different areas of social life, and the concept of fuzzy soft sets was introduced as a fuzzy generalization of soft sets in 2001 by Maji et al [8]. Basic properties of fuzzy soft sets were given in this paper and many scientists such as [9,10,11] improved the works on fuzzy soft sets.

In this paper, some properties of fuzzy soft functions which is mentioned in [9,12] are discussed in detail and we proposed some new properties on image and preimage of certain fuzzy soft sets under a fuzzy soft function.

2 Preliminaries

Let U be an initial universe, E be the set of all possible parameters which are properties of the objects in U, and $\mathscr{P}(U)$ be the set of all subsets of U.

Definition 1. [13] *A fuzzy set A in U is defined by a membership function $\mu_A : U \to [0,1]$ whose membership value $\mu_A(x)$ specifies the degree to which $x \in U$ belongs to the fuzzy set A, for $x \in U$.*

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The symbols $\bigvee_\alpha x_\alpha$ and $\bigwedge_\alpha x_\alpha$ will stand for the supremum and the infimum of x_α' s, respectively.

The family of all fuzzy sets in U will denote by $\mathscr{F}(U)$. If $A,B \in \mathscr{F}(U)$ then some basic fuzzy set operations are given componentwise proposed by Zadeh [1] as follows:

1) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$, for all $x \in U$.
2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$, for all $x \in U$.
3) $C = A \cup B \Leftrightarrow \mu_C(x) = \mu_A(x) \vee \mu_B(x)$, for all $x \in U$.
4) $D = A \cap B \Leftrightarrow \mu_D(x) = \mu_A(x) \wedge \mu_B(x)$, for all $x \in U$.
5) $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$, for all $x \in U$.

Definition 2. [2] *Let A be a subset of E. A pair (F,A) is called a soft set over U where $F : A \to \mathscr{P}(U)$ is a set-valued function.*

As mentioned in [3], a soft set (F,A) can be viewed $(F,A) = \{a = F(a) \mid a \in A\}$ where the symbol "$a = F(a)$" indicates that the approximation for $a \in A$ is $F(a)$.

Definition 3. [4] *For two soft sets (F,A) and (G,B) over a common universe U, we say that (F,A) is a soft subset of (G,B) and is denoted by $(F,A)\tilde{\subset}(G,B)$ if*

(i) $A \subset B$ and,
(ii) $\forall a \in A$, $F(a) \subset G(a)$.

Definition 4. [4] *Two soft sets (F,A) and (G,B) over a common universe U are said soft equal if (F,A) is a soft subset of (G,B), and (G,B) is a soft subset of (F,A).*

Definition 5. [14] *Let (F,A) and (G,B) be two soft sets over the common universe U such that $A \cap B \neq \varnothing$. The soft difference of (F,A) and (G,B) is denoted by $(F,A)\widetilde{-}(G,B)$, and is defined as $(F,A)\widetilde{-}(G,B) = (H,C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) - G(c)$, the difference of the sets $F(c)$ and $G(c)$.*

Definition 6. [4] *The complementof a soft set (F,A) is denoted by $(F,A)^c$ and is defined by $(F,A)^c = (F^c,A)$, where $F^c : A \to \mathscr{P}(U)$ is a mapping given by $F^c(a) = U - F(a)$ for all $a \in A$.*

For the arithmetics in the fuzzy set theory see [15] and for the more set theoretic results in the soft set theory see [2,4,3, 5,11]. The set of all fuzzy sets in U will indicate by $\mathscr{F}(U)$.

Definition 7. [9] *Let $A \subset E$ and $\mathscr{F}(U)$ be the set of all fuzzy sets in U. Then a pair (f,A) is called a fuzzy soft set (fss) over U, where $f : A \to \mathscr{F}(U)$ is a function.*

From the definition, it is clear that $f(a)$ is a fuzzy set in U, for each $a \in A$, and we will denote the membership function of $f(a)$ by $f_a : U \to [0,1]$.

Similar to viewing a soft set, a fuzzy soft set (f,A) can be viewed $(f,A) = \{a = \{u_{f_a(u)} \mid u \in U\} \mid a \in A\}$ where the symbol "$a = \{u_{f_a(u)} \mid u \in U\}$" indicates that the membership degree of the element $u \in U$ is $f_a(u)$ where $f_a : U \to [0,1]$ is the membership function of the fuzzy set $f(a)$ [11].

The family of all fuzzy soft sets over the initial universe U via parameters in E will denote by $\mathscr{FS}(U;E)$

Example 1. Let $U=\{a,b,c\}$ be universe, $E=\{1,2,3\}$ be parameter set and $A=\{1,3\}\subset E$. From Definition 2, $(F,A)=\{1=\{a,b\},\ 3=\{b,c\}\}$ is a soft set over U. From Definition 7, $(f,A)=\{1=\{a_{0.8},b_{0.2}\},3=\{b_{0.4},c_1\}\}$ is a fuzzy soft set over U.

Definition 8. [9] *Let $A,B\subset E$ and $(f,A),(g,B)$ be two fuzzy soft set over a common universe U. We say that (f,A) is fuzzy soft subset of (g,B) and write $(f,A)\widetilde{\subset}(g,B)$ if and only if*

(i) $A\subset B$,
(ii) for each $a\in A$, $f_a(x)\le g_a(x),\forall x\in U$.

Definition 9. [9] *Let $A,B\subset E$. We say that the fuzzy soft sets (f,A) and (g,B) are equal if and only if $(f,A)\widetilde{\subset}(g,B)$ and $(g,B)\widetilde{\subset}(f,A)$.*

Definition 10. *Let (f,A) and (g,B) be two fuzzy soft sets over the common universe U such that $A\cap B\neq\varnothing$. The fuzzy soft difference of (f,A) and (g,B) is denoted by $(f,A)\widetilde{\smile}(g,B)$, and is defined as $(f,A)\widetilde{\smile}(g,B)=(h,C)$, where $C=A\cap B$ and for all $c\in C$, $h_c(x)=f_c(x)\wedge(1-g_c(x)),\forall x\in U$.*

Definition 11. [9] *The complement of a fuzzy soft set (f,A) is the fuzzy soft set (f^c,A), which is denoted by $(f,A)^c$ and where $f^c:A\to\mathscr{F}(U)$ is a fuzzy set-valued function i.e., for each $a\in A$, $f^c(a)$ is a fuzzy set in U, whose membership function $f_a^c(x)=1-f_a(x)$ for all $x\in U$. Here f_a^c is the membership function of $f^c(a)$.*

Definition 12. [8] *Let (f,A) and (g,B) be two fuzzy soft sets over the common universe U. (f,A) AND (g,B), that is a fuzzy soft set over U, is denoted by $(f,A)\wedge(g,B)$, and is defined by $(f,A)\wedge(g,B)=(h,A\times B)$, whose membership function $h_{(a,b)}(x)=f_a(x)\wedge g_b(x)$ for all $(a,b)\in A\times B$ and for all $x\in U$.*

Definition 13. [8] *Let (f,A) and (g,B) be two fuzzy soft sets over the common universe U. (f,A) OR (g,B), that is a fuzzy soft set over U, is denoted by $(f,A)\vee(g,B)$, and is defined by $(f,A)\vee(g,B)=(h,A\times B)$, whose membership function $h_{(a,b)}(x)=f_a(x)\vee g_b(x)$ for all $(a,b)\in A\times B$ and for all $x\in U$.*

3 Fuzzy soft functions

Now, we can define the fuzzy soft function by giving the image and preimage of a fuzzy soft set as the following;

Definition 14. [9,12] *Let U_1, U_2 be initial universes, E_1, E_2 be parameter sets and $\varphi:U_1\to U_2$, $\psi:E_1\to E_2$ be functions. Then the pair (φ,ψ) is said to be a fuzzy soft function from $\mathscr{FS}(U_1;E_1)$ to $\mathscr{FS}(U_2;E_2)$. The image of each $(f,A)\in\mathscr{FS}(U_1;E_1)$ under the fuzzy soft function (φ,ψ) will denoted by $(\varphi,\psi)(f,A)=(\varphi f,\psi(A))$ and the membership function of $(\varphi f)(\beta)$, for each $\beta\in\psi(A)$, is defined as,*

$$(\varphi f)_\beta(y)=\begin{cases}\bigvee\limits_{x\in\varphi^{-1}(y)}\left(\bigvee\limits_{\alpha\in\psi^{-1}(\beta)\cap A} f_\alpha(x)\right) & ,\varphi^{-1}(y)\neq\varnothing,\psi^{-1}(\beta)\cap A\neq\varnothing\\ 0 & ,otherwise\end{cases}$$

for every $y\in U_2$.

The inverse image of each (g,B) in $\mathscr{FS}(U_2;E_2)$ will denoted by $(\varphi,\psi)^{-1}(g,B)=(\varphi^{-1}g,\psi^{-1}(B))$ and the membership function of $(\varphi^{-1}g)(\alpha)$, for each $\alpha\in\psi^{-1}(B)$, is defined as,

$$(\varphi^{-1}g)_\alpha(x) = \begin{cases} g_{\psi(\alpha)}(\varphi(x)) & ,\psi(\alpha) \in B \\ 0 & ,otherwise \end{cases},$$

for every $x \in U_1$.

Example 2. Let followings are given;

$$U_1 = \{a,b,c\}, U_2 = \{x,y,z\}, \varphi = \{(a,x),(b,x),(c,z)\} : U_1 \to U_2,$$

$$E_1 = \{1,2,3,4\}, E_2 = \{5,6,7,8,9\}, \psi = \{(1,5),(2,6),(3,9),(4,9)\} : E_1 \to E_2.$$

To obtain the image of $(f,A) = \{1 = \{a_{0.2}, b_{0.5}, c_{0.1}\}, 4 = \{a_0, b_0, c_1\}\} \in \mathscr{FS}(U_1;E_1)$ with $A = \{1,4\} \subset E_1$ under the fuzzy soft function (φ,ψ), $(\varphi,\psi)(f,A) = (\varphi f, \psi(A))$, we will use these computations:

$$\varphi^{-1}(x) = \{a,b\}, \varphi^{-1}(y) = \varnothing, \varphi^{-1}(z) = \{c\}$$

$$\psi^{-1}(5) = \{1\}, \psi^{-1}(6) = \{2\}, \psi^{-1}(7) = \varnothing, \psi^{-1}(8) = \varnothing, \psi^{-1}(9) = \{3,4\}.$$

Then the membership degrees of all elements in U_2 under the membership functions $(\varphi f)_k$, $k \in \psi(A)$, are as follows;

$$\begin{aligned}(\varphi f)_5(x) &= \bigvee_{t\in\varphi^{-1}(x)} \left(\bigvee_{\alpha\in\psi^{-1}(5)\cap A} f_\alpha(t) \right) = \bigvee_{t\in\{a,b\}} (f_1(t)) = f_1(a) \bigvee f_1(b) \\ &= (0.2) \bigvee (0.5) = 0.5,\end{aligned}$$

$$(\varphi f)_5(y) = \bigvee_{t\in\varphi^{-1}(y)} \left(\bigvee_{\alpha\in\psi^{-1}(5)\cap A} f_\alpha(t) \right) = 0,$$

$$(\varphi f)_5(z) = \bigvee_{t\in\varphi^{-1}(z)} \left(\bigvee_{\alpha\in\psi^{-1}(5)\cap A} f_\alpha(t) \right) = \bigvee_{t\in\{c\}} (f_1(t)) = f_1(c) = 0.1,$$

$$(\varphi f)_6(x) = 0, \qquad (\varphi f)_6(y) = 0, \qquad (\varphi f)_6(z) = 0,$$

$$(\varphi f)_7(x) = 0, \qquad (\varphi f)_7(y) = 0, \qquad (\varphi f)_7(z) = 0,$$

$$(\varphi f)_8(x) = 0, \qquad (\varphi f)_8(y) = 0, \qquad (\varphi f)_8(z) = 0,$$

$$(\varphi f)_9(x) = \bigvee_{t\in\varphi^{-1}(x)} \left(\bigvee_{\alpha\in\psi^{-1}(9)\cap A} f_\alpha(t) \right) = \bigvee_{t\in\{a,b\}} (f_4(t)) = f_4(a) \bigvee f_4(b) = 0,$$

$$(\varphi f)_9(y) = 0,$$

$$(\varphi f)_9(z) = \bigvee_{t\in\varphi^{-1}(z)} \left(\bigvee_{\alpha\in\psi^{-1}(9)\cap A} f_\alpha(t) \right) = \bigvee_{t\in\{c\}} (f_4(t)) = f_4(c) = 1.$$

Hence, we get $(\varphi,\psi)(f,A) = (\varphi f, \psi(A)) = \{5 = \{x_{0.5}, z_{0.1}\}, 9 = \{z_1\}\}$.

The fuzzy soft preimage of $(g,B) = \{5 = \{x_1, y_1, z_{0.1}\}, 8 = \{x_{0.7}, y_{0.1}, z_{0.3}\}, 9 = \{x_{0.1}, y_{0.2}, z_{0.8}\}\} \in \mathscr{FS}(U_2;E_2)$ with

$B=\{5,8,9\}\subset E_2$ under the fuzzy soft function (φ,ψ), $(\varphi,\psi)^{-1}=(\varphi^{-1}g,\psi^{-1}(B))$, is obtained directly as:

$$(\varphi^{-1}g)_1(a)=g_{\psi(1)}(\varphi(a))=g_5(x)=1,$$

$$(\varphi^{-1}g)_1(b)=g_{\psi(1)}(\varphi(b))=g_5(x)=1,$$

$$(\varphi^{-1}g)_1(c)=g_{\psi(1)}(\varphi(c))=g_5(z)=0.1,$$

$$(\varphi^{-1}g)_2(a)=g_{\psi(2)}(\varphi(a))=g_6(x)=0,$$

$$(\varphi^{-1}g)_2(b)=0,$$

$$(\varphi^{-1}g)_2(c)=0,$$

$$(\varphi^{-1}g)_3(a)=g_{\psi(3)}(\varphi(a))=g_9(x)=0.1,$$

$$(\varphi^{-1}g)_3(b)=g_{\psi(3)}(\varphi(b))=g_9(x)=0.1,$$

$$(\varphi^{-1}g)_3(c)=g_{\psi(3)}(\varphi(c))=g_9(z)=0.8,$$

$$(\varphi^{-1}g)_4(a)=g_{\psi(4)}(\varphi(a))=g_9(x)=0.1,$$

$$(\varphi^{-1}g)_4(b)=g_{\psi(4)}(\varphi(b))=g_9(x)=0.1,$$

$$(\varphi^{-1}g)_4(c)=g_{\psi(4)}(\varphi(c))=g_9(z)=0.8,$$

where $\psi^{-1}(B)=\{1,3,4\}$.

Therefore, the preimage of (g,B) is $(\varphi^{-1}g,\psi^{-1}(B))=\{1=\{a_1,b_1,c_{0.1}\},3=\{a_{0.1},b_{0.1},c_{0.8}\},4=\{a_{0.1},b_{0.1},c_{0.8}\}\}$.

In [12], Kharal et al. have folloving results for the image of a fuzzy soft set and for the preimage of a fuzzy soft set:

Theorem 1. [12] *Let (φ,ψ) be a fuzzy soft function between $\mathscr{FS}(U_1;E_1)$ and $\mathscr{FS}(U_2;E_2)$, (f,A) and (g,B) in $\mathscr{FS}(U_1;E_1)$ and $\{(f_k,A_k)\mid k\in K\}$ be a subfamily of $\mathscr{FS}(U_1;E_1)$. Then;*

1) $(\varphi,\psi)(\widetilde{\Phi})=\widetilde{\Phi}$,
2) $(\varphi,\psi)(\widetilde{\mathscr{U}_1})\mathrel{\widetilde{\subseteq}}\widetilde{\mathscr{U}_2}$,
3) $(\varphi,\psi)[(f,A)\mathrel{\widetilde{\cup}}(g,B)]=(\varphi,\psi)(f,A)\mathrel{\widetilde{\cup}}(\varphi,\psi)(g,B)$,
In general, $(\varphi,\psi)\big(\widetilde{\bigcup}_K(f_k,A_k)\big)=\widetilde{\bigcup}_K(\varphi,\psi)(f_k,A_k)$,
4) $(\varphi,\psi)[(f,A)\mathrel{\widetilde{\cap}}(g,B)]\mathrel{\widetilde{\subseteq}}(\varphi,\psi)(f,A)\mathrel{\widetilde{\cap}}(\varphi,\psi)(g,B)$,
In general, $(\varphi,\psi)\big(\widetilde{\bigcap}_K(f_k,A_k)\big)\mathrel{\widetilde{\subseteq}}\widetilde{\bigcap}_K(\varphi,\psi)(f_k,A_k)$,
5) If $(f,A)\mathrel{\widetilde{\subseteq}}(g,B)$, then $(\varphi,\psi)(f,A)\mathrel{\widetilde{\subseteq}}(\varphi,\psi)(g,B)$.

Theorem 2. [12] *Let (φ,ψ) be a fuzzy soft function between $\mathscr{FS}(U_1;E_1)$ and $\mathscr{FS}(U_2;E_2)$, (f,A) and (g,B) in $\mathscr{FS}(U_2;E_2)$ and $\{(f_k,A_k)\mid k\in K\}$ be a subfamily of $\mathscr{FS}(U_2;E_2)$. Then;*

1) $(\varphi,\psi)^{-1}(\widetilde{\Phi})=\widetilde{\Phi}$,
2) $(\varphi,\psi)^{-1}(\widetilde{\mathscr{U}_2})=\widetilde{\mathscr{U}_1}$,
3) $(\varphi,\psi)^{-1}[(f,A)\mathrel{\widetilde{\cup}}(g,B)]=(\varphi,\psi)^{-1}(f,A)\mathrel{\widetilde{\cup}}(\varphi,\psi)^{-1}(g,B)$,
In general, $(\varphi,\psi)^{-1}\big(\widetilde{\bigcup}_K(f_k,A_k)\big)=\widetilde{\bigcup}_K(\varphi,\psi)^{-1}(f_k,A_k)$,

4) $(\varphi,\psi)^{-1}[(f,A)\,\widetilde{\sqcap}\,(g,B)] = (\varphi,\psi)^{-1}(f,A)\,\widetilde{\sqcap}\,(\varphi,\psi)^{-1}(g,B)$,
In general, $(\varphi,\psi)^{-1}\left(\widetilde{\bigcap}_K(f_k,A_k)\right) = \widetilde{\bigcap}_K(\varphi,\psi)^{-1}(f_k,A_k)$,
5) If $(f,A)\,\widetilde{\sqsubseteq}\,(g,B)$, *then* $(\varphi,\psi)^{-1}(f,A)\,\widetilde{\sqsubseteq}\,(\varphi,\psi)^{-1}(g,B)$.

They also showed in [12] by giving examples that the inequalities (2), (4) and implication (5) in Theorem 1 and the implication (5) in Theorem 2 can not be reversible in general.

Now some new results on fuzzy soft functions are as follows;

Theorem 3. *Let* (φ,ψ) *be a fuzzy soft function from* $\mathscr{FS}(U_1;E_1)$ *to* $\mathscr{FS}(U_2;E_2)$*. Then for any* $(f,A)\in\mathscr{FS}(U_1;E_1)$*, we have*

$$[(\varphi,\psi)(f,A)]^c\,\widetilde{\sqsubseteq}\,(\varphi,\psi)(f,A)^c.$$

Proof. Since $[(\varphi,\psi)(f,A)]^c = (\varphi f,\psi(A))^c = ((\varphi f)^c,\psi(A))$ and $(\varphi,\psi)(f,A)^c = (\varphi,\psi)(f^c,A) = (\varphi f^c,\psi(A))$ we have for each $\beta\in\psi A$ that

$$(\varphi f)^c_\beta(y) = 1-(\varphi f)_\beta(y) = 1-\left[\bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap A} f_\alpha(x)\right)\right]$$

and

$$\begin{aligned}(\varphi f^c)_\beta(y) &= \bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap A} f^c_\alpha(x)\right)\\ &= \bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap A} (1-f_\alpha(x))\right)\\ &= 1-\left[\bigwedge_{x\in\varphi^{-1}(y)}\left(\bigwedge_{\alpha\in\psi^{-1}(\beta)\cap A} f_\alpha(x)\right)\right],\end{aligned}$$

for all $y\in U$. Hence $(\varphi f)^c_\beta(y)\le(\varphi f^c)_\beta(y)$. So, the claim is true.

Following example shows that for $(f,A),(g,B)\in\mathscr{FS}(U_1;E_1)$ neither

$$[(\varphi,\psi)(f,A)\widetilde{\sqcup}(\varphi,\psi)(g,B)]\widetilde{\sqsubseteq}(\varphi,\psi)\,[(f,A)\widetilde{\sqcup}(g,B)]$$

nor

$$(\varphi,\psi)\,[(f,A)\widetilde{\sqcup}(g,B)]\,\widetilde{\sqsubseteq}[(\varphi,\psi)(f,A)\widetilde{\sqcup}(\varphi,\psi)(g,B)]\,.$$

Example 3. Let $U_1=\{a,b,c,d\}, U_2=\{x,y,z\}$ be universes and $\varphi: U_1\to U_2$ be a function such that $\varphi(a)=x,\varphi(b)=x,\varphi(c)=y,\varphi(d)=y$. $E_1=\{1,2,3,4,5\}$, $E_2=\{6,7,8,9,10,11\}$ be parameter sets and $\psi: E_1\to E_2$ be a function such that $\psi(1)=6,\psi(2)=6,\psi(3)=8,\psi(4)=10,\psi(5)=10$.

Let

$$(f,A)=\{1=\{a_0,b_1,c_{0.3},d_{0.5}\},3=\{a_{0.2},b_{0.5},c_{0.7},d_{0.2}\},5=\{a_1,b_{0.1},c_{0.1},d_{0.8}\}\}$$

and

$$(g,B)=\{2=\{a_{0.5},b_{0.8},c_{0.7},d_{0.6}\},4=\{a_{0.7},b_{0.5},c_{0.8},d_{0.3}\},5=\{a_{0.1},b_{0.7},c_{0.6},d_{0.3}\}\}$$

be two fuzzy soft sets in $\mathscr{F}\mathscr{S}(U_1;E_1)$, where $A=\{1,3,5\}$ and $B=\{2,4,5\}$ are subsets of E_1. Then from Definition 10 and Definition 14 we have the image of the fuzzy soft difference of (f,A) and (g,B) under the fuzzy soft function (φ,ψ) as;

$$(\varphi,\psi)[(f,A)\widetilde{\smile}(g,B)]=\{10=\{x_{0.9},y_{0.7},z_0\}\}=(k\text{ (say)},\{10\}),$$

and we have the fuzzy soft difference of $(\varphi,\psi)(f,A)$ and $(\varphi,\psi)(g,B)$ as;

$$\begin{aligned}(\varphi,\psi)(f,A)\widetilde{\smile}(\varphi,\psi)(g,B)&=\{6=\{x_{0.2},y_{0.3},z_0\},10=\{x_{0.3},y_{0.2},z_0\}\}\\&=(d\text{ (say)},\{6,10\}).\end{aligned}$$

Note that, $\{10\}\subset\{6,10\}$ which shows that $(\varphi,\psi)(f,A)\widetilde{\smile}(\varphi,\psi)(g,B)$ is not a subset of $(\varphi,\psi)[(f,A)\widetilde{\smile}(g,B)]$ and $d_{10}(x)=0.3\le 0.9=k_{10}(x)$ which shows that $(\varphi,\psi)[(f,A)\widetilde{\smile}(g,B)]$ is not a subset of $(\varphi,\psi)(f,A)\widetilde{\smile}(\varphi,\psi)(g,B)$.

Theorem 4. *Let (f,A), (g,B) be in $\mathscr{F}\mathscr{S}(U_1;E_1)$ and (φ,ψ) be a fuzzy soft function from $\mathscr{F}\mathscr{S}(U_1;E_1)$ to $\mathscr{F}\mathscr{S}(U_2;E_2)$. If ψ is one to one function and $g_\alpha(x)$ is constant, for all $x\in\varphi^{-1}(y)$, $y\in U_2$ and for all $\alpha\in\psi^{-1}(\beta)\cap(A\cap B)$, $\beta\in\psi(A\cap B)$. Then*

$$(\varphi,\psi)\left[(f,A)\widetilde{\smile}(g,B)\right]=\left[(\varphi,\psi)(f,A)\right]\widetilde{\smile}\left[(\varphi,\psi)(g,B)\right].$$

Proof. From Definition 10, we have $(f,A)\widetilde{\smile}(g,B)=(h,A\cap B)$, where, for each $d\in A\cap B$, $h_d(x)=f_d(x)\wedge(1-g_d(x)),\forall x\in U_1$ and from Definition 14, we have $(\varphi,\psi)\left[(f,A)\widetilde{\smile}(g,B)\right]=(\varphi h,\psi(A\cap B))$ where, for each $\beta\in\psi(A\cap B)$,

$$\begin{aligned}(\varphi h)_\beta(y)&=\bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap(A\cap B)}h_\alpha(x)\right)\\&=\bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap(A\cap B)}[f_\alpha(x)\wedge(1-g_\alpha(x)]\right)\\&=\left(\bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap(A\cap B)}f_\alpha(x)\right)\right)\\&\wedge\left(1-\bigwedge_{x\in\varphi^{-1}(y)}\left(\bigwedge_{\alpha\in\psi^{-1}(\beta)\cap(A\cap B)}g_\alpha(x)\right)\right),\end{aligned}$$

for all $y\in U_2$. Similarly,

$$\left[(\varphi,\psi)(f,A)\widetilde{\smile}(\varphi,\psi)(g,B)\right]=(\varphi f,\psi(A))\widetilde{\smile}(\varphi g,\psi(B))=(k,\psi(A)\cap\psi(B))$$

where, for each $\beta \in \psi(A) \cap \psi(B) = \psi(A \cap B)$ (since ψ one to one),

$$\begin{aligned} k_\beta(y) &= (\varphi f)_\beta(y) \wedge [1 - (\varphi g)_\beta(y)] \\ &= \left(\bigvee_{x \in \varphi^{-1}(y)} \left(\bigvee_{\alpha \in \psi^{-1}(\beta) \cap (A \cap B)} f_\alpha(x) \right) \right) \\ &\wedge \left(1 - \bigvee_{x \in \varphi^{-1}(y)} \left(\bigvee_{\alpha \in \psi^{-1}(\beta) \cap (A \cap B)} g_\alpha(x) \right) \right) \end{aligned}$$

for all $y \in U_2$.

From hypothesis of theorem, we have for each $\beta \in \psi(A \cap B)$, $(\varphi h)_\beta(y) = k_\beta(y)$ for all $y \in U_2$.

Together with, $\psi(A \cap B) = \psi(A) \cap \psi(B)$, consequently, we write

$$(\varphi, \psi)[(f, A) \widetilde{\smile} (g, B)] = [(\varphi, \psi)(f, A)] \widetilde{\smile} [(\varphi, \psi)(g, B)]$$

This completes the proof.

Theorem 5. *Let (φ, ψ) be a fuzzy soft function from $\mathscr{FS}(U_1; E_1)$ to $\mathscr{FS}(U_2; E_2)$ and $(f, A), (g, B) \in \mathscr{FS}(U_2; E_2)$. Then we have*

$$[(\varphi, \psi)^{-1}(f, A) \widetilde{\smile} (\varphi, \psi)^{-1}(g, B)] = (\varphi, \psi)^{-1}[(f, A) \widetilde{\smile} (g, B)] \,.$$

Proof. From Definition 10 and Definition 14, we have

$$\begin{aligned} [(\varphi, \psi)^{-1}(f, A)] \widetilde{\smile} [(\varphi, \psi)^{-1}(g, B)] &= (\varphi^{-1} f, \psi^{-1}(A)) \widetilde{\smile} (\varphi^{-1} g, \psi^{-1}(B)) \\ &= (h, \psi^{-1}(A) \cap \psi^{-1}(B)) \end{aligned}$$

where, for each $\alpha \in \psi^{-1}(A) \cap \psi^{-1}(B)$,

$$h_\alpha(x) = (\varphi^{-1} f)_\alpha(x) \wedge \left[1 - (\varphi^{-1} g)_\alpha(x)\right] = f_{\psi(\alpha)}(\varphi(x)) \wedge \left(1 - g_{\psi(\alpha)}(\varphi(x))\right),$$

for all $x \in U_1$. And, we have

$$(\varphi, \psi)^{-1}[(f, A) \widetilde{\smile} (g, B)] = (\varphi, \psi)^{-1}(k, A \cap B) = (\varphi^{-1} k, \psi^{-1}(A \cap B))$$

where, for each $d \in A \cap B$, $k_d(y) = f_d(y) \wedge (1 - g_d(y))$, for all $y \in U_2$. So, for each $\alpha \in \psi^{-1}(A \cap B)$ we get

$$(\varphi^{-1} k)_\alpha(x) = k_{\psi(\alpha)}(\varphi(x)) = f_{\psi(\alpha)}(\varphi(x)) \wedge \left(1 - g_{\psi(\alpha)}(\varphi(x))\right),$$

for all $x \in U_1$. From computations above and the fact that $\psi^{-1}(A \cap B) = \psi^{-1}(A) \cap \psi^{-1}(B)$ for any function on crisp sets, we have

$$[(\varphi, \psi)^{-1}(f, A) \widetilde{\smile} (\varphi, \psi)^{-1}(g, B)] = (\varphi, \psi)^{-1}[(f, A) \widetilde{\smile} (g, B)].$$

Definition 15. [12] *A fuzzy soft set* (f,E) *over* U *is said to be an* absolute fuzzy soft set *and denoted by* $\widetilde{\mathscr{U}}$ *if and only if for each* $e \in E$*,* $f_e = \widetilde{1}$*, where* $\widetilde{1}$ *is the membership function of the absolute fuzzy set over* U*, which takes value* 1 *for all* x *in* U*.*

Lemma 1. *Let* U *be an initial universe and* E *be a parameter set. Then for any* $(f,A) \in \mathscr{F}\mathscr{S}(U;E)$ *we have* $\widetilde{\mathscr{U}} \widetilde{\smile} (f,A) = (f,A)^c$*.*

Proof. From Definition 10, we have, $\widetilde{\mathscr{U}} \widetilde{\smile} (f,A) = (h, A \cap E) = (h,A)$, where each $a \in A$, $h_a(x) = 1 \wedge (1 - f_a(x)) = 1 - f_a(x)$, for all $x \in U$.

From Definition 11 we write $\widetilde{\mathscr{U}} \widetilde{\smile} (f,A) = (f,A)^c$.

Corollary 1. *Let* (φ, ψ) *be a fuzzy soft function from* $\mathscr{F}\mathscr{S}(U_1;E_1)$ *to* $\mathscr{F}\mathscr{S}(U_2;E_2)$*. Then for any* $(f,A) \in \mathscr{F}\mathscr{S}(U_2;E_2)$*, we have*

$$[(\varphi, \psi)^{-1}(f,A)]^c = (\varphi, \psi)^{-1}(f,A)^c \ .$$

Proof. If we take $\widetilde{\mathscr{U}_2}$ instead of (f,A) in the Theorem 5. the proof is obvious from Lemma 1.

Theorem 6. *Let* (φ, ψ) *be a fuzzy soft function from* $\mathscr{F}\mathscr{S}(U_1;E_1)$ *to* $\mathscr{F}\mathscr{S}(U_2;E_2)$*. Then for any* $(f,A) \in \mathscr{F}\mathscr{S}(U_1;E_1)$*, we have*

$$(f,A) \ \widetilde{\subset} \ (\varphi, \psi)^{-1}[(\varphi, \psi)(f,A)] \ .$$

Proof. From Definition 14 we have

$$(\varphi, \psi)^{-1}[(\varphi, \psi)(f,A)] = (\varphi, \psi)^{-1}(\varphi f, \psi(A)) = (\varphi^{-1}(\varphi f), \psi^{-1}(\psi(A))).$$

It is obvious that $A \subset \psi^{-1}(\psi(A))$ and we can compute for each $\alpha \in \psi^{-1}(\psi(A))$ that

$$(\varphi^{-1}(\varphi f))_{\alpha}(x) = (\varphi f)_{\psi(\alpha)}(\varphi(x)) = \bigvee_{m \in \varphi^{-1}(\varphi(x))} \left(\bigvee_{\gamma \in \psi^{-1}(\psi(\alpha)) \cap A} f_{\gamma}(m) \right),$$

for all $x \in U_1$.

Hence from the computation above and the fact that $\bigvee_{x \in A} x \leq \bigvee_{x \in B} x$ for $A \subset B \subset [0,1]$, we can write $(f,A) \ \widetilde{\subset} \ (\varphi, \psi)^{-1}[(\varphi, \psi)(f,A)]$.

Example 4. Let $U_1 = \{a,b,c\}$, $U_2 = \{x,y,z\}$ be initial universes and $\varphi : U_1 \to U_2$ be a function such that $\varphi(a) = x$, $\varphi(b) = x$, $\varphi(c) = z$. Let $E_1 = \{1,2,3,4\}$, $E_2 = \{5,6,7,8\}$ be parameter sets, and $\psi : E_1 \to E_2$ be a function such that $\psi(1) = 5$, $\psi(2) = 5$, $\psi(3) = 6$, $\psi(4) = 8$.

Let $(f,A) = \{1 = \{a_{0.2}, b_{0.3}, c_1\}, 3 = \{a_{0.7}, b_{0.6}, c_0\}\}$ be a fuzzy soft set in $\mathscr{F}\mathscr{S}(U_1;E_1)$, where $A = \{1,3\} \subset E_1$. Then, after several computation, we have $(\varphi,\psi)^{-1}[(\varphi,\psi)(f,A)] = \{1 = \{a_{0.3}, b_{0.3}, c_1\}, 2 = \{a_{0.3}, b_{0.3}, c_1\}, 3 = \{a_{0.7}, b_{0.7}, c_0\}\}$ which shows that, the implication in Theorem 6 is not reversible in general and note that if φ and ψ are one to one functions then the implication will reverse, i.e., $(f,A) = (\varphi,\psi)^{-1}[(\varphi,\psi)(f,A)]$.

Theorem 7. *Let (φ,ψ) be a fuzzy soft function from $\mathscr{F}\mathscr{S}(U_1;E_1)$ to $\mathscr{F}\mathscr{S}(U_2;E_2)$. Then for any $(g,B) \in \mathscr{F}\mathscr{S}(U_2;E_2)$, we have*

$$(\varphi,\psi)[(\varphi,\psi)^{-1}(g,B)] \,\tilde{\subset}\, (g,B)\ .$$

Proof. From Definition 14 we have

$$(\varphi,\psi)[(\varphi,\psi)^{-1}(g,B)] = (\varphi,\psi)(\varphi^{-1}g, \psi^{-1}(B)) = (\varphi(\varphi^{-1}g), \psi(\psi^{-1}(B))).$$

For each $\beta \in \psi(\psi^{-1}(B))$,

$$\begin{aligned}
\left(\varphi(\varphi^{-1}g)\right)_\beta(y) &= \bigvee_{x\in\varphi^{-1}(y)} \left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap\psi^{-1}B} (\varphi^{-1}g)(x) \right) \\
&= \bigvee_{x\in\varphi^{-1}(y)} \left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap\psi^{-1}B} g_{\psi(\alpha)}(\varphi(x)) \right) \\
&= \bigvee_{x\in\varphi^{-1}(y)} \left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap\psi^{-1}B} g_\beta(\varphi(x)) \right) \\
&= \bigvee_{x\in\varphi^{-1}(y)} g_\beta(\varphi(x)) \\
&= \bigvee_{x\in\varphi^{-1}(y)} g_\beta(y) \\
&= g_\beta(y),
\end{aligned}$$

for all $y \in U_2$. Hence we can write

$$(\varphi,\psi)[(\varphi,\psi)^{-1}(g,B)] \,\tilde{\subset}\, (g,B)$$

since $\psi(\psi^{-1}(B)) \subset B$ for all $B \subset E_2$.

Example 5. Let the situation be as in Example 3, and $(g,B) = \{5 = \{x_{0.2}, y_{0.3}, z_1\}, 8 = \{x_0, y_{0.5}, z_{0.2}\}\} \in \mathscr{F}\mathscr{S}(U_2;E_2)$ where $B = \{5,8\} \subset E_2$. Then, we have the image of (g,B) under the fuzzy soft function (φ,ψ) as;

$$(\varphi,\psi)[(\varphi,\psi)^{-1}(g,b)] = \{5 = \{x_{0.2}, y_0, z_1\}, 8 = \{x_0, y_0, z_{0.2}\}\}$$

which shows that in general the implication in Theorem 7 is not reversible and note that we have $(\varphi,\psi)[(\varphi,\psi)^{-1}(g,B)] = (g,B)$ when ψ is surjective.

Definition 16. *Let U_1,U_2,U_3 and E_1,E_2,E_3 be universes and parameter sets, respectively, and $(\varphi,\psi) : \mathscr{F}\mathscr{S}(U_1;E_1) \to \mathscr{F}\mathscr{S}(U_2;E_2)$ and $(\sigma,\varsigma) : \mathscr{F}\mathscr{S}(U_2;E_2) \to \mathscr{F}\mathscr{S}(U_3;E_3)$ be fuzzy soft functions. Then the* *composition of fuzzy soft functions* (φ,ψ) *and* (σ,ς) *is fuzzy soft function from* $\mathscr{F}\mathscr{S}(U_1;E_1)$ *to* $\mathscr{F}\mathscr{S}(U_3;E_3)$ *which is defined and denoted as*

$$(\sigma,\varsigma)\circ(\varphi,\psi)=(\sigma\circ\varphi,\varsigma\circ\psi)\,.$$

Theorem 8. *Let* (φ,ψ) *be a fuzzy soft function from* $\mathscr{F}\mathscr{S}(U_1;E_1)$ *to* $\mathscr{F}\mathscr{S}(U_2;E_2)$ *and* (σ,ς) *be a fuzzy soft function from* $\mathscr{F}\mathscr{S}(U_2;E_2)$ *to* $\mathscr{F}\mathscr{S}(U_3;E_3)$*. Then for all* $(f,A)\in\mathscr{F}\mathscr{S}(U_3;E_3)$*, we have*

$$[(\sigma,\varsigma)\circ(\varphi,\psi)]^{-1}(f,A)=(\varphi,\psi)^{-1}[(\sigma,\varsigma)^{-1}(f,A)]\,.$$

Proof. We have Definition 14 and 15 that,

$$[(\sigma,\varsigma)\circ(\varphi,\psi)]^{-1}(f,A)=(\sigma\circ\varphi,\varsigma\circ\psi)^{-1}(f,A)=((\sigma\circ\varphi)^{-1}f,(\varsigma\circ\psi)^{-1}(A))$$

and for each $\alpha\in(\varsigma\circ\psi)^{-1}(A)$,

$$[(\sigma\circ\varphi)^{-1}f]_\alpha(x)=f_{(\varsigma\circ\psi)(\alpha)}((\sigma\circ\varphi)(x))=f_{\varsigma(\psi(\alpha))}(\sigma(\varphi(x))),$$

for all $x\in U_1$. Similarly, we have,

$$(\varphi,\psi)^{-1}[(\sigma,\varsigma)^{-1}(f,A)]=(\varphi,\psi)^{-1}(\sigma^{-1}f,\varsigma^{-1}(A))=(\varphi^{-1}(\sigma^{-1}f),\psi^{-1}(\varsigma^{-1}(A)))$$

and, for each $\alpha\in\psi^{-1}(\varsigma^{-1}(A))$,

$$(\varphi^{-1}(\sigma^{-1}f))_\alpha(x)=(\sigma^{-1}f)_{\psi(\alpha)}(\varphi(x))=f_{\varsigma(\psi(x))}(\sigma(\varphi(x))),$$

for all $x\in U_1$. Since $(\varsigma\circ\psi)^{-1}(A)=\psi^{-1}(\varsigma^{-1}(A))$ for any $A\subset E_3$, these computations imply $[(\sigma,\varsigma)\circ(\varphi,\psi)]^{-1}(f,A)=(\varphi,\psi)^{-1}[(\sigma,\varsigma)^{-1}(f,A)]$.

Theorem 9. *Let* (φ,ψ) *be a fuzzy soft function from* $\mathscr{F}\mathscr{S}(U_1;E_1)$ *to* $\mathscr{F}\mathscr{S}(U_2;E_2)$ *and* (f,A) *and* (g,B) *be fuzzy soft sets over* U_1*. Then*

$$(\varphi,\psi^*)((f,A)\wedge(g,B))=(\varphi,\psi)(f,A)\wedge(\varphi,\psi)(g,B)$$

where $\psi^*:E_1\times E_1\to E_2\times E_2$ *such that* $\psi^*(e_1,e_2)=(\psi(e_1),\psi(e_2))$ *for all* $e_1,e_2\in E_1$.

Proof. From Definition 12 and Definition 14, we have

$$(\varphi,\psi^*)((f,A)\wedge(g,B))=(\varphi,\psi^*)(h,A\times B)=(\varphi h,\psi^*(A\times B)).$$

Since $\psi^*(A\times B)=\psi(A)\times\psi(B)$, we obtain $(\varphi h,\psi^*(A\times B))=(\varphi h,\psi(A)\times\psi(B))$. Its membership function is

$$\begin{aligned}(\varphi h)_{(c,d)}(y) &= \bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{(a,b)\in(\psi^*)^{-1}((c,d))\cap A\times B} h_{(a,b)}(x)\right)\\ &= \bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{(a,b)\in(\psi^*)^{-1}((c,d))\cap A\times B} f_a(x)\wedge g_b(x)\right)\\ &= \left[\bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{a\in\psi^{-1}(c)\cap A} f_a(x)\right)\right]\wedge\left[\bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{b\in\psi^{-1}(d)\cap B} g_b(x)\right)\right]\\ &= (\varphi f)_c(y)\wedge(\varphi g)_d(y). \qquad (1)\end{aligned}$$

On the other hand, we have

$$(\varphi,\psi)(f,A)\wedge(\varphi,\psi)(g,B)=(\varphi f,\psi(A))\wedge(\varphi g,\psi(B)=(k,\psi(A)\times\psi(B)).$$

For all $(c,d)\in\psi(A)\times\psi(B)=\psi^*(A\times B)$ and for all $y\in U_2$, the membership function of $(k,\psi(A)\times\psi(B))$ is as follows;

$$k_{(c,d)}(y)=(\varphi f)_c(y)\wedge(\varphi g)_d(y). \qquad (2)$$

Thus, we obtain $(\varphi,\psi^*)((f,A)\wedge(g,B))=(\varphi,\psi)(f,A)\wedge(\varphi,\psi)(g,B)$ from 1 and 2.

The proof of following theorem can be obtained with a similar way with Theorem 9.

Theorem 10. *Let (φ,ψ) be a fuzzy soft function from $\mathscr{F}\mathscr{S}(U_1;E_1)$ to $\mathscr{F}\mathscr{S}(U_2;E_2)$ and (f,A) and (g,B) be fuzzy soft sets over U_1. Then*

$$(\varphi,\psi^*)((f,A)\vee(g,B))=(\varphi,\psi)(f,A)\vee(\varphi,\psi)(g,B).$$

Theorem 11. *Let (φ,ψ) be a fuzzy soft function from $\mathscr{F}\mathscr{S}(U_1;E_1)$ to $\mathscr{F}\mathscr{S}(U_2;E_2)$ and (f,A) and (g,B) be fuzzy soft sets over U_2. Then*

$$(\varphi,\psi^*)^{-1}((f,A)\wedge(g,B))=(\varphi,\psi)^{-1}(f,A)\wedge(\varphi,\psi)^{-1}(g,B).$$

Proof. From Definition 13 and Definition 14, we have

$$(\varphi,\psi^*)^{-1}((f,A)\wedge(g,B))=(\varphi,\psi^*)^{-1}(h,A\times B)=(\varphi^{-1}h,(\psi^*)^{-1}(A\times B)).$$

Since $(\psi^*)^{-1}(A\times B)=\psi^{-1}(A)\times\psi^{-1}(B)$, we obtain

$$(\varphi^{-1}h,(\psi^*)^{-1}(A\times B))=(\varphi^{-1}h,\psi^{-1}(A)\times\psi^{-1}(B)).$$

The following computation gives the membership function of the fuzzy set $(\varphi^{-1}h)((a,b))$, for all $(a,b) \in (\psi^*)^{-1}(A\times B) = \psi^{-1}(A)\times\psi^{-1}(B)$. For all $x \in U_1$,

$$\begin{aligned}(\varphi^{-1}g)_{(a,b)}(x) &= h_{\psi^*((a,b))}(\varphi(x))\\ &= h_{(\psi(a),\psi(b))}(\varphi(x))\\ &= f_{\psi(a)}(\varphi(x)) \wedge g_{\psi(b)}(\varphi(x))\\ &= (\varphi^{-1}f)_a(x) \wedge (\varphi^{-1}g)_b(x).\end{aligned} \tag{3}$$

On the other hand, we have

$$\begin{aligned}(\varphi,\psi)^{-1}(f,A)\wedge(\varphi,\psi)^{-1}(g,B) &= (\varphi^{-1}f,\psi^{-1}(A))\wedge(\varphi^{-1}g,\psi^{-1}(B)\\ &= (k,\psi^{-1}(A)\times\psi^{-1}(B)).\end{aligned}$$

For all $(a,b) \in \psi^{-1}(A)\times\psi^{-1}(B) = (\psi^*)^{-1}(A\times B)$ and for all $x \in U_1$ the membership function of a fuzzy set $k((a,b))$;

$$k_{(a,b)}(x) = (\varphi^{-1}f)_a(x)\wedge(\varphi^{-1}g)_b(x). \tag{4}$$

Thus we obtain,

$$(\varphi,\psi^*)^{-1}((f,A)\wedge(g,B)) = (\varphi,\psi)^{-1}(f,A)\wedge(\varphi,\psi)^{-1}(g,B)$$

from 3 and 4.

Also similar to above, we obtain following theorem and proof of this theorem is done in a similar manner.

Theorem 12. *Let (φ,ψ) be a fuzzy soft function from $\mathscr{FS}(U_1;E_1)$ to $\mathscr{FS}(U_2;E_2)$ and (f,A) and (g,B) be fuzzy soft sets over U_2. Then*

$$(\varphi,\psi^*)^{-1}((f,A)\vee(g,B)) = (\varphi,\psi)^{-1}(f,A)\vee(\varphi,\psi)^{-1}(g,B).$$

Definition 17. [9] *Let U_1, U_2 be initial universes, E_1, E_2 be parameter sets and $\varphi : U_1 \to U_2$, $\psi : E_1 \to E_2$ be functions. We called that (φ,ψ) is injective (surjective) if φ and ψ is injective (surjective respectively).*

We will call, (φ,ψ) is bijective if φ and ψ are bijective.

Definition 18. *Let $1_U : U \to U$ and $1_E : E \to E$.The fuzzy soft function*

$$(1_U,1_E) : \mathscr{FS}(U;E) \to \mathscr{FS}(U;E)$$

is called fuzzy soft identity function and denoted by $1_{\mathscr{FS}(U;E)}$.

Definition 19. *Let U_1, U_2 be initial universes, E_1, E_2 be parameter sets and $\varphi : U_1 \to U_2$, $\psi : E_1 \to E_2$ be functions. We called that (σ,ς) is inverse fuzzy soft function of (φ,ψ) and denoted by $(\sigma,\varsigma) = (\varphi,\psi)^{-1} = (\varphi^{-1},\psi^{-1})$, such that*

$$(\sigma,\varsigma)\circ(\varphi,\psi) = (1_{U_1}, 1_{E_1})$$

and

$$(\varphi,\psi)\circ(\sigma,\varsigma) = (1_{U_2}, 1_{E_2}).$$

Remark. The reader can easily note that, if (φ,ψ) and (σ,ς) are bijective, then their composition $(\varphi,\psi)\circ(\sigma,\varsigma)$ is also bijective.

Theorem 13. *Let (φ,ψ) be injective fuzzy soft function i.e. φ and ψ is injective and let (f,A) and (g,B) be fuzzy soft set over U_1. If $(\varphi,\psi)(f,A) = (\varphi,\psi)(g,B)$, then $(f,A) = (g,B)$.*

Proof. From Definition 14, we have $(\varphi f, \psi(A)) = (\varphi g, \psi(B))$. So, we obtain $\varphi f = \varphi g$ and $\psi(A) = \psi(B)$. Since ψ is injective, so we obtain $A = B$. On the other hand, for all $\beta \in \psi(A) = \psi(B)$ and for all $y \in U_2$ we have

$$\begin{aligned}(\varphi f)_\beta(y) &= \bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap A} f_\alpha(x)\right)\\ &= \bigvee_{x\in\varphi^{-1}(y)}\left(\bigvee_{\alpha\in\psi^{-1}(\beta)\cap B} g_\alpha(x)\right) = (\varphi g)_\beta(y).\end{aligned}$$

Since $A = B$, φ is injective and, from the above equality, we obtain $f = g$. Thus, $(f,A) = (g,B)$.

In [16], Qin and Hong defined the concept of soft equality between soft sets. The following definition can be given as a generalization of definition of soft equality for fuzzy soft sets.

Definition 20. *Let (f,A) and (g,B) be two fuzzy soft sets over common universe U. Then,*

(i) (f,A) is called null fuzzy soft equal to (g,B), denoted by $(f,A) \approx_{fs} (g,B)$, if for all $e \in A\cup B$ and for all $x \in U$, $e \in A\cap B$ implies $f_e(x) = g_e(x)$, $e \in A - B$ implies $f_e(x) = 0$, and $e \in B - A$ implies $g_e(x) = 0$.

(ii)(f,A) is called whole fuzzy soft equal to (g,B), denoted by $(f,A) \approx^{fs} (g,B)$, if for all $e \in A\cup B$ and for all $x \in U$, $e \in A\cap B$ implies $f_e(x) = g_e(x)$, $e \in A - B$ implies $f_e(x) = 1$, and $e \in B - A$ implies $g_e(x) = 1$.

Theorem 14. *Let (φ,ψ) be a fuzzy soft function from $\mathscr{FS}(U_1;E_1)$ to $\mathscr{FS}(U_2;E_2)$ and (f,A) and (g,B) be fuzzy soft sets over U_1. If $(f,A) \approx_{fs} (g,B)$ and ψ is injective, then*

$$(\varphi,\psi)(f,A) \approx_{fs} (\varphi,\psi)(g,B).$$

Proof. From Definition 14, we have $(\varphi,\psi)(f,A) = (\varphi f, \psi(A))$ and $(\varphi,\psi)(g,B) = (\varphi g, \psi(B))$. So, if $e' \in \psi(A)\cup\psi(B) = \psi(A\cup B)$, then there is $e \in A\cup B$ such that $\psi(e) = e'$. Since $(f,A) \approx_{fs} (g,B)$, if $e \in A\cap B$, then $f_e(x) = g_e(x)$ for all $x \in U_1$.

Therefore, for $e' \in \psi(A\cap B) = \psi(A)\cap\psi(B)$ and for all $y \in U_2$,

$$\begin{aligned}(\varphi f)_{e'}(y) &= \bigvee_{x\in\varphi^{-1}(y)} \left(\bigvee_{e\in\psi^{-1}(e')\cap(A\cap B)} f_e(x) \right) \\ &= \bigvee_{x\in\varphi^{-1}(y)} \left(\bigvee_{e\in\psi^{-1}(e')\cap(A\cap B)} g_e(x) \right) \\ &= (\varphi g)_{e'}(y).\end{aligned}$$

Thus we obtain that $e' \in \psi(A)\cap\psi(B)$ implies $(\varphi f)_{e'}(y) = (\varphi g)_{e'}(y)$.

Let $e' \in \psi(A)-\psi(B)$. Since $\psi(A)-\psi(B) = \psi(A-B)$ we have for all $y \in U_2$ that

$$(\varphi f)_{e'}(y) = \bigvee_{x\in\varphi^{-1}(y)} \left(\bigvee_{e\in\psi^{-1}(e')\cap(A-B)} f_e(x) \right) = 0.$$

Similarly, for $e' \in \psi(B)-\psi(A)$ and for all $y \in U_2$, we have

$$(\varphi g)_{e'}(y) = \bigvee_{x\in\varphi^{-1}(y)} \left(\bigvee_{e\in\psi^{-1}(e')\cap(B-A)} g_e(x) \right) = 0.$$

Consequently, $(\varphi,\psi)(f,A) \approx_{fs} (\varphi,\psi)(g,B)$.

Theorem 15. *Let (φ,ψ) be a fuzzy soft function from $\mathscr{FS}(U_1;E_1)$ to $\mathscr{FS}(U_2;E_2)$ and (f,A) and (g,B) be fuzzy soft sets over U_1. If $(f,A) \approx^{fs} (g,B)$ and ψ is injective, then*

$$(\varphi,\psi)(f,A) \approx^{fs} (\varphi,\psi)(g,B).$$

Proof. Similar to proof of Theorem 14.

We obtain following theorems for inverse image of fuzzy soft sets which is fuzzy soft equal under a fuzzy soft function.

Theorem 16. *Let (φ,ψ) be a fuzzy soft function from $\mathscr{FS}(U_1;E_1)$ to $\mathscr{FS}(U_2;E_2)$ and (f,A) and (g,B) be fuzzy soft sets over U_2. If $(f,A) \approx_{fs} (g,B)$, then*

$$(\varphi,\psi)^{-1}(f,A) \approx_{fs} (\varphi,\psi)^{-1}(g,B).$$

Proof. We have $(\varphi,\psi)^{-1}(f,A) = (\varphi^{-1}f, \psi^{-1}(A))$ and $(\varphi,\psi)^{-1}(g,B) = (\varphi^{-1}g, \psi^{-1}(B))$ by Definition 14. So, if $e \in \psi^{-1}(A)\cup\psi^{-1}(B) = \psi^{-1}(A\cup B)$, then $\psi(e) \in A\cup B$. Since $(f,A) \approx_{fs} (g,B)$, if $\psi(e) \in A\cap B$, then $f_{\psi(e)}(y) = g_{\psi(e)}(y)$ for all $y \in U_2$. Therefore, for $e \in \psi^{-1}(A) \cap \psi^{-1}(B) = \psi^{-1}(A \cap B)$ and for all $x \in U_1$,

$$(\varphi^{-1} f)_e(x) = f_{\psi(e)}(\varphi(x)) = g_{\psi(e)}(\varphi(x)) = (\varphi^{-1} g)_e(x).$$

Thus we obtain $e \in \psi^{-1}(A) \cap \psi^{-1}(B)$ implies $(\varphi^{-1} f)_e(x) = \varphi^{-1} g)_e(x)$.

On the other hand, if $e \in \psi^{-1}(A) - \psi^{-1}(B) = \psi^{-1}(A - B)$, then we obtain

$$(\varphi^{-1} f)_e(x) = f_{\psi(e)}(\varphi(x)) = 0$$

for all $x \in U_1$. Similarly, if $e \in \psi^{-1}(B) - \psi^{-1}(A) = \psi^{-1}(B - A)$, then we obtain

$$(\varphi^{-1} g)_e(x) = g_{\psi(e)}(\varphi(x)) = 0$$

for all $x \in U_1$.

Consequently, $(\varphi, \psi)^{-1}(f, A) \approx_{fs} (\varphi, \psi)^{-1}(g, B)$.

Theorem 17. *Let (φ, ψ) be a fuzzy soft function from $\mathscr{F}\mathscr{S}(U_1; E_1)$ to $\mathscr{F}\mathscr{S}(U_2; E_2)$ and (f, A) and (g, B) be fuzzy soft sets over U_2. If $(f, A) \approx^{fs} (g, B)$, then*

$$(\varphi, \psi)^{-1}(f, A) \approx^{fs} (\varphi, \psi)^{-1}(g, B).$$

Proof. Similar to proof of Theorem 16.

4 Conclusions

Since both fuzzy set theory and soft set theory deal with the problems including vagueness, uncertainties etc., fuzzy soft set theory has a huge potential to solve these kinds of problems from each part of real life. So to contribute having a way to get a solution for these kinds of problems, one can need the concept of fuzzy soft function between two fuzzy soft sets, since a fuzzy soft function can be thought as a relation between two fuzzy soft sets. In this paper we have studied some functional properties of fuzzy soft functions for the fuzzy soft sets. We hope that this results will help the researchers to improve the fuzzy soft set theory.

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