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Application of different analytical methods to equation system of Bodewadt's fixed disc and rotating stream

Sajjad Sattari, Mahsa Jahani and Mofid Gorji-Bandpy

Department of Mechanical Engineering, Babol University of Technology, P.O. Box 484, Babol, Iran.

Received: 6 December 2014, Revised: 16 December 2014, Accepted: 20 July 2015
Published online: 30 September 2015

Abstract: The similarity transform for the steady three-dimensional problem of a rotating stream over a fixed disc gives a system of nonlinear ordinary differential equations which are analytically solved by homotopy perturbation, homotopy analysis and variational iteration methods. The analytic solutions of the system of nonlinear ordinary differential equations are constructed in the series form except for the variational iteration method in which the solutions are obtained at the end of each iteration. The convergence of the obtained series solutions from homotopy analysis method (HAM) is carefully analyzed. Comparison of the results of the applied method with the numerical solution is also provided in this paper.

Keywords: Bodewadt's equation system, system of nonlinear differential equation, homotopy analysis method, homotopy perturbation method, variational iteration method, rotating stream.

1 Introduction

Most scientific problems and phenomena are modeled by nonlinear ordinary or partial differential equations. Some of them are solved using numerical methods and some are solved using analytic methods of perturbation [1]-[2]. In numerical methods, stability and convergence should be our main concerns so as to avoid divergence or undesirable results. In analytic perturbation methods, we should introduce a small parameter into the equation. Therefore, various new methods have recently presented some techniques to remove the small parameter, such as the Adomian decomposition method [3]-[7], the homotopy analysis method [8]-[11], the homotopy perturbation method [12]-[24] and the variational iteration method [25]-[36].

The variational iteration method (VIM) was first introduced by He and was successfully applied to autonomous ordinary differential equations [37], to Helmholtz equations [38], to nonlinear differential equations of fractional order [39], and in many other fields. The use of variational iteration method (VIM), the differential transforms method and the Adomian decomposition method (ADM) for solving differential equations was introduced in [40]. The homotopy-perturbation method (HPM) was also first introduced by He. This method does not depend on a small parameter. Homotopy perturbation method (HPM) was successfully applied to nonlinear oscillators with discontinuities [41], heat radiation equations [42]-[43], etc.

In 1992, Liao [44] employed the basic ideas of homotopy the basic ideas of homotopy in topology to propose a general analytic method for nonlinear problems, namely the homotopy analysis method (HAM), [45]-[50]. Based on homotopy of topology, the validity of homotopy analysis method (HAM) is independent of the existence of the small parameter. The homotopy analysis method (HAM) always provides us with a family of solution expressions containing an auxiliary

* Corresponding author e-mail: Sajjad.Sattari@gmail.com

parameter which can properly chosen to give accurate solutions.

The main goal of the present article is to find the totally analytic solution for the problem of rotating stream fixed disc which is very similar to Von-karman rotating problem with the same variables (F, G, H and P) as function of a non-dimensional independent variable L and with an important modification (which might be called a Â ”trickÂ”): in order for the radial momentum equation to balance with rotating stream, there must be a positive nonzero radial pressure gradient.

Consider the steady rotating flow with a constant angular velocity, about the axis $r = 0$. All three velocity components v_r, v_θ and v_z would be involved. However they would be independent because of the symmetry.

2 Mathematical formulation

The momentum Navier-Stokes and continuity equations are as follows:

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0 \tag{1a}$$

$$\rho\left(v_r\frac{\partial(rv_r)}{\partial r}-\frac{v_\theta^2}{r}+\frac{\partial(v_r)}{\partial z}\right)=-\frac{1}{\rho}+v\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial(v_r)}{\partial r}+\frac{{v_r}^2}{r}+\frac{\partial^2 v_r}{\partial z^2}\right) \tag{1b}$$

$$v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta v_r}{r}+v_z\frac{\partial v_\theta}{\partial z}=v\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial(v_\theta)}{\partial r}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{v_\theta}{r^2}\right) \tag{1c}$$

$$v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{\rho}\frac{\partial\rho}{\partial r}+v\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{\partial^2 v_z}{\partial z^2}\right) \tag{1d}$$

with the boundary condition:

$$\begin{aligned} &at\ z=0: \quad v_r=v_\theta=v_z=0 \quad p=cte=0\\ &at\ z\to\infty: \ v_r=0 \qquad v_\theta=1 \end{aligned} \tag{2}$$

With $\eta = z\sqrt{\frac{w}{v}}$:

$$v_r = r\omega F(\eta) \tag{3a}$$

$$v_\theta = r\omega G(\eta) \tag{3b}$$

$$v_z = \sqrt{v\omega}H(\eta) \tag{3c}$$

$$P = \rho\omega v P(\eta)+\rho r \tag{3d}$$

and with substitution to the main equations:

$$F''+G^2-F^2-F'h-1=0 \tag{4a}$$

$$H'+2F=0 \tag{4b}$$

$$G''-2FG-HG'=0 \tag{4c}$$

$$P'-2FH+2F'=0 \tag{4d}$$

where prime denotes differentiation with respect Ã§, the boundary conditions are:

$$\begin{aligned} &at\ \eta = 0: \quad F = G = H = P = 0 \\ &at\ \eta \to \infty \quad F = 0 \quad G = 1 \end{aligned} \tag{5}$$

In order to use the methods we have to find the proper form of initial conditions [51]. To do so we introduce six new functions as follows:

$$Y_1 = H,\ Y_2 = F',\ Y_3 = F,\ Y_4 = G',\ Y_5 = G \tag{6}$$

So the equations are reduced by one order of derivative:

$$Y_2'(\eta) = -Y_5^2 + Y_3^2 + Y_2.Y_1 + 1 \tag{7a}$$

$$Y_1' + 2Y_3 = 0 \tag{7b}$$

$$Y_3' = Y_2 \tag{7c}$$

$$Y_4' = 2Y_3.Y_5 + Y_1.Y_4 \tag{7d}$$

$$Y_5' = Y_4 \tag{7e}$$

By use a numerical method we can easily obtain:

$$\begin{aligned} G'(0) &= 0.77289 \\ F'(0) &= -0.94197 \end{aligned} \tag{8}$$

3 Basic idea of homotopy perturbation method

To illustrate the basic ideas of the new method, we consider the following nonlinear differential equation:

$$A(u) - F(r) = 0, \qquad r \in \Omega \tag{9}$$

Subject to the following condition:

$$B\left(u, \frac{\partial u}{\partial n} = 0\right) \tag{10}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and γ is the boundary of the domain Ω

Generally speaking, the operator A can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Eq.(9) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \tag{11}$$

By the homotopy technique, we construct a homotopy $V(r,p) : \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad P \in [0,1], \quad r \in \Omega.$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \quad P \in [0,1], \quad r \in \Omega. \tag{12}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (9), which satisfies the boundary conditions. Obviously, from Eq. (12) we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{13}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{14}$$

Thus the changing process of q of $V(r,p)$ from $u_0(r)$ to $u(r)$ In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; besides, it can be considered as a small parameter for $0 \le q \le 1$ So, it is very quite right to assume that the soloutions of Eq. (11) can be expressed as:

$$v = v_0 + v_1 q + v_2 q^2 + \dots \tag{15}$$

The approximate soloution of Eq. (9) can therefore be clearly obtained:

$$u = \lim_{q \to 1}(v) = v_0 + v_1 + v_2 + \dots \tag{16}$$

4 Basic idea of variational iteration method

To clarify the basic ideas of variational iteration method (VIM), we consider the following differential equation:

$$L(u) + N(v) = g(x) \tag{17}$$

where L is a linear differential operator, N a nonlinear analytic operator, and $g(x)$ an inhomogeneous term. According to the VIM, we can constract a correction funvtional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \gamma[\ell\{u_n(\tau)\} + N\{\tilde{u}_n(\tau)\} - g(\tau)]d\tau \tag{18}$$

where γ is a general Lagrange multiplier [52], which can be identify optimally via the variational theory [53]-[54], the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation [54], i.e., $\delta\tilde{u}_n = 0$.

5 Basic ideas of homotopy analysis method

Consider the following differential equation:

$$N[u(\tau)] = 0 \tag{19}$$

where N is a nonlinear operator, τ denotes an independent variable, and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. By means of generalizing the traditional homotopy method, Liao [55] constructed the so-called zero-order deformation equation as:

$$(1-q)L[\Phi(\tau;q) - u_0(\tau)] = q\lambda A(\tau)N[\Phi(\tau;q)] \tag{20}$$

where $q \in [0,1]$ is the embedding parameter, λ a non-zero auxiliary parameter, $A(\tau) \neq 0$ an auxiliary function, L an auxiliary linear operator, $u_0(\tau)$ an initial guess of $u(\tau)$ and $\Phi(\tau;q)$ is an unknown function. It is important to have enough freedom to choose auxiliary unknowns in homotopy Analysis method (HAM).

Obviously, when $q = 0$ and $q = 1$, it holds:

$$\Phi(\tau;0) = u_0(\tau), \quad \Phi(\tau;1) = u(\tau) \tag{21}$$

Thus, as q increases from 0 to 1, The solution $\Phi(\tau;q)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding by Taylor series with respect to q, we have:

$$\Phi(\tau;q) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)q^m \tag{22}$$

Where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \Phi(\tau,q)}{\partial q^m}|q = 0 \tag{23}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter λ, and the auxiliary function are so properly chosen, the series Eq. (22) converges at $q = 1$, then we have:

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau) \tag{24}$$

which must be one of solution for the original nonlinear equation, as proved by Liao [5]. As $\lambda = -1$ and $A(\tau) = 1$, Eq. (19) then becomes:

$$(1-q)L[\Phi(\tau;q) - u_0(\tau)] + qN[\Phi(\tau;q)] = 0 \tag{25}$$

This is mostly used in HPM, whereas the solution can be obtained direstly without using Taylor series. According to Eq. (20), the governing equation can be deduced from the zero-deformation equation. The vector is defined as:

$$\tilde{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\} \tag{26}$$

Differentiating Eq. (20) m times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing them by $m!$, we will have the so-called m^{th} order deformation equation as:

$$L[u_m(\tau) - X_m u_{m-1}(\tau)] = \lambda A(\tau) R_m(\tilde{u}_{m-1}) \tag{27}$$

Where

$$R_m(u_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\Phi(\tau;q)}{\partial q^{m-1}}|q = 0 \tag{28}$$

and

$$X_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases} \tag{29}$$

It should be emphasized that $u_m(\tau)$ for $m > 1$ is governed by the linear Eq. (27) with the linear boundary conditions coming from the original problem, which can be easily solved using a symbolic computation software.

6 Application of homotopy perturbation method (HPM)

To investigate the explicit and totally analytic solutions of Eq. (4) by using HPM, we begin with defining the linear and nonlinear operators for each equation as follows:

$$L_F\{f\} = f''$$
$$N_F\{f\} = g^2 - f^2 - f'h - 1 \tag{30}$$
$$L_H\{h\} = h'$$
$$N_H\{h\} = 2f \tag{31}$$
$$L_G\{g\} = g''$$
$$N_G\{g\} = -2fg - hg' \tag{32}$$
$$L_P\{p\} = p'$$
$$N_P\{p\} = -2fg + 2f' \tag{33}$$

Where

$$F = \lim_{q\to 1} f = f_0 + f_1 + f_2 + \dots$$
$$G = \lim_{q\to 1} g = g_0 + g_1 + g_2 + \dots$$
$$H = \lim_{q\to 1} h = h_0 + h_1 + h_2 + \dots$$
$$P = \lim_{q\to 1} p = p_0 + p_1 + p_2 + \dots \tag{34}$$

Here in this article for a better convergence we add and subtract some linear terms to each equation:

$$F'' + 2F - 2F + G^2 - F^2 - F'H - 1 = 0$$
$$L_F\{f\} = f'' + 2f$$
$$N_F\{f\} = g^2 - f^2 - f'h - 1 - 2f \tag{35}$$
$$H' + 2H - 2H + 2F = 0$$
$$L_H\{h\} = h' + 2h$$
$$N_H\{h\} = 2f - 2h \tag{36}$$
$$G'' + 2G - 2G + g^2 - 2FG - HG' = 0$$
$$L_G\{g\} = g'' + 2g$$
$$N_G\{g\} = -2fg - hg' - 2g \tag{37}$$
$$P' + 2P - 2P - 2FH + 2F' = 0$$
$$L_P\{p\} = P' + 2P$$
$$N_P\{p\} = -2fg + 2f' - 2p \tag{38}$$

We can now construct the homotopy functions as follows:

$$H_1(f,g,h,q) = (1-p)(f'' + 2f) + q(f'' + g^2 - f^2 - f'h - 1) \tag{39a}$$

$$H_2(f,h,q) = (1-p)(h' + 2h) + q(h' + 2f') \tag{40a}$$

$$H_3(f.g.h.q) = (1-p)(g + 2g) + q(g + 2fq + hg') \tag{40b}$$

$$H_4(f,g,h,q) = (1-p)(p' + 2f) = q \tag{40c}$$

with defining the series form offunction in six terms:

$$f = \sum_{m=0}^{6} f_m q^m \tag{41a}$$

$$g = \sum_{m=0}^{6} g_m q^m \tag{41b}$$

$$h = \sum_{m=0}^{6} h_m q^m \tag{41c}$$

$$p = \sum_{m=0}^{6} p_m q^m \tag{41d}$$

Setting all $L(u_0)$ to zero, the solutions are:

$$f_0 = -\frac{94197}{200000}\sin(\sqrt{2}\eta)\sqrt{2}$$
$$g_0 = \frac{77289}{200000}\sin(\sqrt{2}\eta)\sqrt{2}$$
$$h_0 = 0$$
$$p_0 = 0$$

According to Eq. (34) the solutions for F, G, H and P can be found.

7 Application of variational iteration method (VIM)

According to Eq. (18), we can construct correction functional as follows:

$$F_{n+1} = F_n + \int_0^n \gamma_1 (F_{n_{\tau,\tau}} + G_n(\tau)^2 - F_n^2 - F_{n_\tau} H_n - 1)\mathrm{d}\tau \tag{42a}$$

$$H_{n+1} = H_n + \int_0^n \gamma_2 (H_{n_\tau} + 2F_n(\tau))\mathrm{d}\tau \tag{42b}$$

$$G_{n+1} = G_n + \int_0^n \gamma_3 (G_{n_{\tau,\tau}} - 2F_n(\tau)G_n - G_{n_\tau} H_n)\mathrm{d}\tau \tag{42c}$$

$$P_{n+1} = P_n + \int_0^n \gamma_4 (P_{n_\tau} - 2F_n(\tau)H_n(\tau) + 2F_{n_\tau})\mathrm{d}\tau \tag{42d}$$

The Lagrange multipliers are obtained as follows:

$$\begin{aligned}\gamma_1 &= \tau-\eta\\ \gamma_2 &= -1\\ \gamma_3 &= \tau-\eta\\ \gamma_4 &= -1\end{aligned} \quad (43)$$

We set the initial values as follows:

$$\begin{aligned}F_0 &= -0.94197\eta\\ H_0 &= 0\\ G_0 &= -0.77289\eta\\ P_0 &= 0\end{aligned} \quad (44)$$

Solving the Eq. (42) with MAPLE for 8 iterations gives the solutions:

$$\begin{aligned}F_1 &= -0.94197\eta+0.5\eta^2+0.02416237740\eta^4\\ H_1 &= 0.94197\eta^2\\ G_1 &= 0.77289\eta-0.121339865\eta^4\\ P_1 &= 1.88294\eta\end{aligned} \quad (45)$$

8 Application of homotopy analysis method (HAM)

First we construct the so-called mth order deformation equations as follows:

$$L_f\{f_m(\eta)-X_m f_{m-1}(\eta)\} = \lambda A_f R_m^f(f_{m-1}) \quad (46a)$$

$$L_g\{f_m(\eta)-X_m g_{m-1}(\eta)\} = \lambda A_g R_m^g(g_{m-1}) \quad (46b)$$

$$L_h\{h_m(\eta)-X_m h_{m-1}(\eta)\} = \lambda A_h R_m^h(h_{m-1}) \quad (46c)$$

$$L_p\{p_m(\eta)-X_m p_{m-1}(\eta)\} = \lambda A_p R_m^p(p_{m-1}) \quad (46d)$$

Where

$$f(\eta) = f_0(\eta)+\sum_{m=1}^{\infty} f_m(\eta) \quad (47a)$$

$$g(\eta) = g_0(\eta)+\sum_{m=1}^{\infty} g_m(\eta) \quad (47b)$$

$$h(\eta) = h_0(\eta)+\sum_{m=1}^{\infty} h_m(\eta) \quad (47c)$$

$$p(\eta) = p_0(\eta)+\sum_{m=1}^{\infty} p_m(\eta) \quad (47d)$$

Due to the physical of the problem that is a decaying fluctuating function by η, we assume the solution in the form below:

$$F(\eta) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_{m,n} \mathrm{e}^{-\mathrm{m}\eta} \sin(n\eta) \tag{48a}$$

$$G(\eta) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_{m,n} \mathrm{e}^{-\mathrm{m}\eta} \cos(n\eta) \tag{48b}$$

$$H(\eta) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} c_{m,n} \mathrm{e}^{-\mathrm{m}\eta} \sin(n\eta) \tag{48c}$$

$$P(\eta) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} d_{m,n} \mathrm{e}^{-\mathrm{m}\eta} \sin(n\eta) \tag{48d}$$

where $a_{m,n}, b_{m,n}, c_{m,n}$ and $d_{m,n}$ called rule of solution expression, guides us to the selection of an auxiliary function which is denoted by $A(\tau)$. According to Eq. (4) and Eqs. (48), we choose the linear operator as the following terms:

$$L_f\{f(\eta;q)\} = \frac{d^2 f(\eta;q)}{d\eta^2} + 3\frac{df(\eta;q)}{d\eta} + 2f(\eta;q) \tag{49a}$$

$$L_g\{g(\eta;q)\} = \frac{d^2 g(\eta;q)}{d\eta^2} + 4\frac{dg(\eta;q)}{d\eta} + 5g(\eta;q) \tag{49b}$$

$$L_h\{h(\eta;q)\} = \frac{dh(\eta;q)}{d\eta} + h(\eta;q) \tag{49c}$$

$$L_p\{p(\eta;q)\} = \frac{dp(\eta;q)}{d\eta} \tag{49d}$$

with the properties:

$$L_f\{C_1 \mathrm{e}^{-2\mathrm{x}} + C_2 \mathrm{e}^{-\mathrm{x}}\} = 0 \tag{50a}$$

$$L_g\{C_3 \mathrm{e}^{-2\mathrm{x}} \sin(x) + C_4 \mathrm{e}^{-2\mathrm{x}} \cos(x)\} = 0 \tag{50b}$$

$$L_h\{C_5 \mathrm{e}^{-\mathrm{x}}\} = 0 \tag{50c}$$

$$L_p\{C_6\} = 0 \tag{50d}$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are constants to be determined through the initial conditions. Now we verify the nonlinear operators:

$$N_f\{f(\eta;q), g(\eta;q), h(\eta;q)\} = \frac{d^2 f(\eta;q)}{d\eta^2} + g(\eta;q)^2 - f(\eta;q)^2 - \frac{df(\eta;q)}{d\eta} h(\eta;q) - 1 \tag{51a}$$

$$N_g\{f(\eta;q), g(\eta;q), h(\eta;q)\} = \frac{d^2 g(\eta;q)}{d\eta^2} - 2f(\eta;q)g(\eta;q) - \frac{dg(\eta;q)}{d\eta} h(\eta;q) \tag{51b}$$

$$N_h\{f(\eta;q), h(\eta;q)\} = \frac{dh(\eta;q)}{d\eta} + 2f(\eta;q) \tag{51c}$$

$$N_p\{f(\eta;q), g(\eta;q), h(\eta;q)\} = \frac{dp(\eta;q)}{d\eta} - 2f(\eta;q)h(\eta;q) + \frac{2df(\eta;q)}{d\eta} \tag{51d}$$

According to Eq. (4) and Eq. (48), the initial guess should be in the form:

$$f_0 = -0.94\sigma(\eta)e^{-\eta}$$
$$g_0 = 1 - e^{-\eta}\cos(\eta) - 0.23\sin(\eta)e^{-\eta}$$
$$h_0 = 0.5 - 0.5e^{-\eta}\cos(\eta) - 0.5\sin(\eta)e^{-\eta}$$
$$p_0 = 1.88\sin(\eta)e^{-\eta} \tag{52}$$

They satisfy the initial condition and the rule of solution expression. We have also:

$$R_m^f(f_{m-1}, g_{m-1}, h_{m-1}) = \frac{d^2 f_{m-1}(\eta)}{d\eta^2} - \left(\sum_{j=0}^{m-1} f_j(\eta) f_{m-1-j}(\eta)\right)$$
$$+\left(\sum_{j=0}^{m-1} g_j(\eta) g_{m-1-j}(\eta)\right) - \left(\sum_{j=0}^{m-1} h_j(\eta)\left(\frac{d}{dz} f_{m-1-j}(\eta)\right)\right) - 1 - X_m$$
$$(m; n\ are\ coefficients) \tag{53a}$$

$$R_m^g(f_{m-1}, g_{m-1}, h_{m-1}) = \frac{d^2 g_{m-1}(\eta)}{d\eta^2} - 2\left(\sum_{j=0}^{m-1} f_j(\eta) g_{m-1-j}(\eta)\right)$$
$$-\left(\sum_{j=0}^{m-1} h_j(\eta)\left(\frac{dg_{m-1-j}(\eta)}{dz}\right)\right) \tag{53b}$$

$$R_m^h(f_{m-1}, h_{m-1}) = \frac{dh_{m-1}(\eta)}{d(\eta)} + 2f_{m-1}(\eta) \tag{53c}$$

$$R_m^p(f_{m-1}, p_{m-1}, h_{m-1}) = \frac{dp_{m-1}(\eta)}{d(\eta)} + 2\left(\frac{df_{m-1}(\eta)}{d(\eta)}\right)$$
$$-2\left(\sum_{j=0}^{m-1} f_j(\eta) h_{m-1-j}(\eta)\right) \tag{53d}$$

Now the solution of mth order deformation equations can be obtained by applying inverse linear operator to both sides of the above equations.
Due to the rule of the solution expression we choose the auxiliary functions as follows:

$$A_f(\eta) = 1, \quad A_g(\eta) = 1, A_h(\eta) = 1, \quad A_p(\eta) = 1, \tag{54}$$

By choosing a value for m the solution for f, g, h and p can be obtained by using Eqs. (47).

9 Result and comparison between the methods

The following figures illustrate the behavior of the four parameters in the three methods of homotopy perturbation method (HPM), variational iteration method (VIM) and homotopy analysis method (HAM), respectively. The results clearly show that the homotopy perturbation method (HPM) is capable of solving a large class of nonlinear equations with rapid convergent successive approximations but with some restraining assumptions (the adding and subtracting some linear terms to the linear part of the equations for instance). Variational iteration method (VIM), on the other hand, does not need any excess assumptions. It also reduces the amount of calculations considerably. The homotopy analysis method (HAM) gives us accurate results and also the ability of controlling the solutions by introducing the parameter Î and the auxiliary function A.

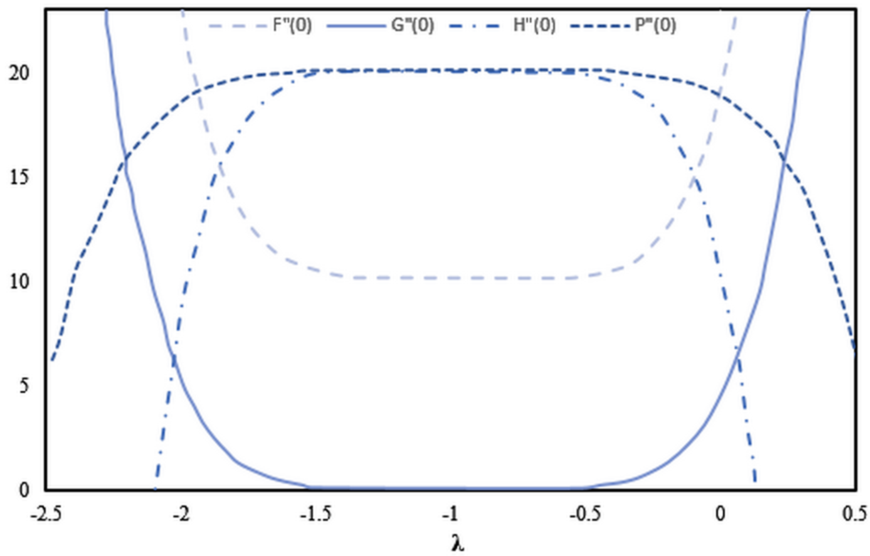

Fig. 1: λ -curve for $F''(0), G''(0), H''(0)$ and $P''(0)$

Table 1: Some summarized results for $F(\eta)$

η	VIM	HPM	HAM	Numerical Solution	Errore of VIM	Errore of HPM	Errore of HAM
0.1	-0.08920182	-0.08920182	-0.08920181	-0.089201825	1.34E-08	1.09E-07	1.31E-07
0.2	-0.16846881	-0.16846881	-0.16846881	-0.168468805	-4.77E-09	-1.60E-08	-1.01E-08
0.3	-0.23795814	-0.23795814	-0.23795814	-0.237958144	1.46E-09	2.12E-08	2.54E-08
0.4	-0.29791327	-0.29791332	-0.29791332	-0.29791327	9.63E-09	-1.77E-07	-1.74E-07
0.5	-0.34865000	-0.34865035	-0.34865035	-0.348650008	2.45E-08	-9.85E-07	-9.82E-07
0.6	-0.39054444	-0.39054603	-0.39054603	-0.390544448	3.55E-08	-4.05E-06	-4.04E-06
0.7	-0.42402228	-0.42402758	-0.42402758	-0.424022291	2.28E-08	-1.25E-05	-1.25E-05
0.8	-0.4495942	-0.44956304	-0.44956302	-0.449549426	5.19E-09	-3.03E-5	-3.02E-05
0.9	-0.46762349	-0.4676502	-0.4676502	-0.467623511	5.06E-08	-5.71E-05	-5.71E-05
1	-0.47876623	-0.47880139	-0.47880139	-0.478766249	5.11E-08	-7.34E-05	-7.34E-05

Table 2: Some summarized results for $G(\eta)$

η	VIM	HPM	HAM	Numerical Solution	Errore of VIM	Errore of HPM	Errore of HAM
0.1	0.07728319	0.077283176	0.077283174	0.077283185	-7.42E-08	1.09E-07	1.35E-07
0.2	0.154489124	0.154489124	0.154489121	0.154489118	-3.49E-08	-4.01E-08	-2.07E-08
0.3	0.231437388	0.231437378	0.231437378	0.231437381	-2.91E-08	1.33E-08	1.41E-08
0.4	0.307861036	0.307860969	0.307860968	0.307861031	-1.77E-08	1.99E-07	2.03E-07
0.5	0.383432705	0.383432111	0.383432112	0.383432705	1.05E-10	1.55E-06	1.55E-06
0.6	0.457787543	0.457784067	0.457784066	0.457787544	22.11E-09	7.60E-06	7.60E-06
0.7	0.530542759	0.530527571	0.530527571	0.530542742	-3.26E-08	2.86E-05	2.86E-05
0.8	0.601313866	0.601260082	0.601260082	0.6013313848	-2.89E-08	8.94E-05	8.94E-05
0.9	0.669727845	0.669566226	0.669566223	0.669727857	1.73E-08	2.41E-04	2.41E-04
1	0.735433525	0.735007992	0.735007992	0.73543353	5.89E-09	5.78E-04	5.78E-04

10 Conclusion

In this paper, three kinds of analytical methods, called the homotopy perturbation method (HPM), the variational iteration method (VIM) and the homotopy analysis method (HAM) have been successfully applied to find explicit

Table 3: Some summarized results for $H(\eta)$

η	VIM	HPM	HAM	Numerical Solution	Errore of VIM	Errore of HPM	Errore of HAM
0.1	0.009086561	0.009086564	0.009086554	0.009086566	6.19E-07	2.49E-07	1.35E-06
0.2	0.035018181	0.035018175	0.035018175	0.035018192	3.38E-07	5.02E-07	5.06E-07
0.3	0.075822049	0.075821989	0.075821988	0.075822066	2.27E-07	1.02E-06	1.03E-06
0.4	0.129565663	0.129565207	0.125652	0.129565684	1.62E-07	3.68E-06	3.74E-06
0.5	0.194372663	0.194370439	0.194370438	0.194372683	1.01E-07	1.15E-05	1.15E-05
0.6	0.268436067	0.268428101	0.2684281	0.268436093	9.98E-08	2.98E-05	2.98E-05
0.7	0.350029242	0.350006088	0.350006087	0.350029302	1.72E-07	6.63E-05	6.63E-05
0.8	0.437514862	0.437456824	0.437456824	0.437514952	2.07E-07	1.33E-04	1.33E-04
0.9	0.529352086	0.52922054	0.529220541	0.529352151	1.23E-07	2.49E-04	2.49E-04
1	0.624102131	0.623823802	0.623823802	0.624102232	1.62E-07	4.46E-04	4.46E-04

Table 4: Some summarized results for $P(\eta)$

η	VIM	HPM	HAM	Numerical Solution	Errore of VIM	Errore of HPM	Errore of HAM
0.1	0.178362365	0.17836236	0.17836226	0.178362393	1.58E-07	1.85E-07	7.54E-07
0.2	0.336324475	0.336324347	0.336324373	0.336324528	1.58E-07	4.57E-07	4.60E-07
0.3	0.473041795	0.473040436	0.473040434	0.473041859	1.35E-07	3.01E-06	3.01E-06
0.4	0.587432904	0.58742459	0.58742459	0.587432966	1.05E-07	1.43E-05	1.43E-05
0.5	0.678409633	0.67837687	0.67837686	0.678409676	6.21E-08	4.84E-05	4.84E-05
0.6	0.745059908	0.744961714	0.744961712	0.745059949	5.52E-08	1.32E-04	1.32E-04
0.7	0.786784326	0.786538668	0.786538668	0.786784366	5.1E-08	3.12E-04	3.12E-4
0.8	0.80338922	0.80284583	0.80284563	0.803389215	-7.16E-09	6.76E-04	6.77E-04
0.9	0.795140161	0.794037295	0.794037295	0.795140191	3.78E-08	1.39E-03	1.39E-03
1	0.762780716	0.760678244	0.760678242	0.76278071	-8.70E-09	2.76E-03	2.76E-.3

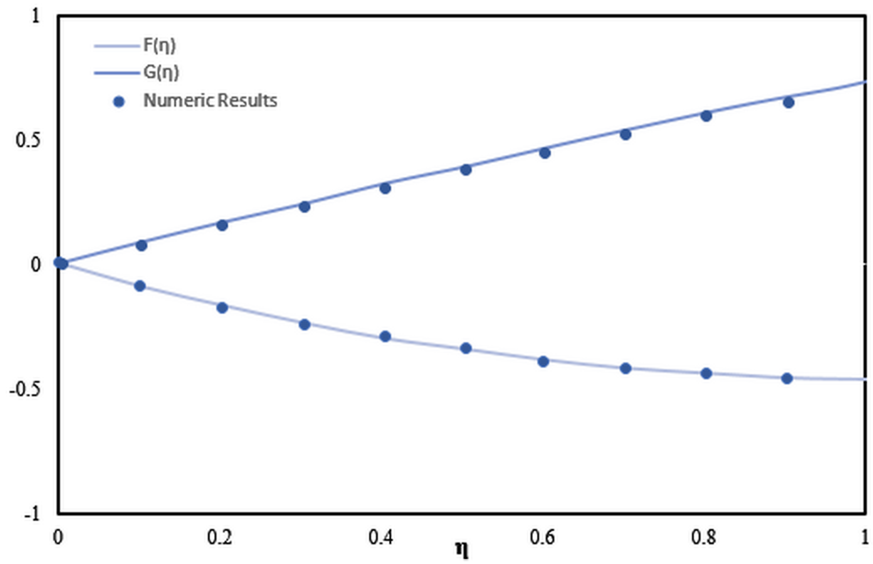

Fig. 2: Comparison between the numerical results and the results obtained by VIM and HAM for $F(\eta)$ and $G(\eta)$ with $\lambda = -1$ and $m = 9$

solution of the equation system of Bodewat's.

The results clearly show that all methods are extremely effective for solving nonlinear differential equations. The

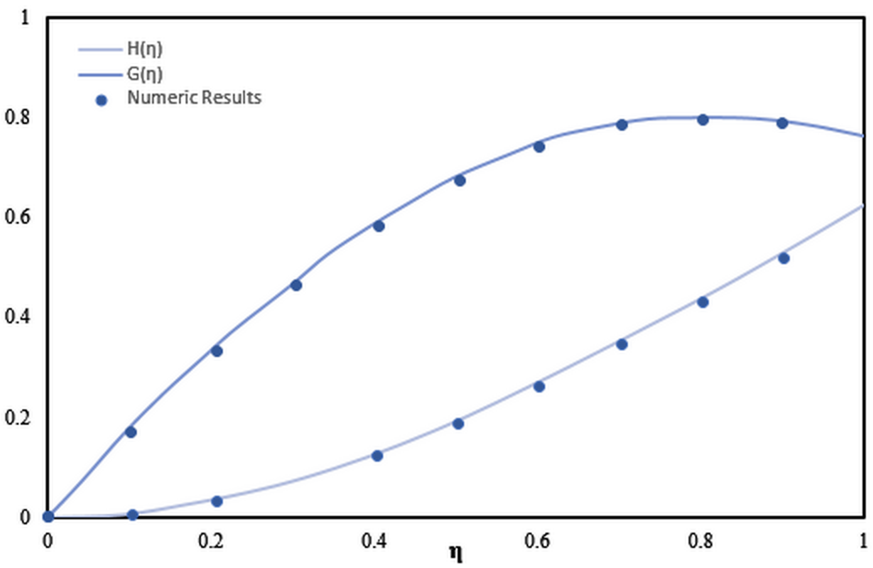

Fig. 3: Comparison between the numerical results and the results obtained by VIM and HAM for $H(\eta)$ and $P(\eta)$ with $\lambda = -1$ and $m = 9$

homotopy perturbation method (HPM) and variational iteration method (VIM) are easy to use and are more rapid convergent. However the flexibility of using them is low and the ability to control and manage-the-process of solution is difficult. They are also hard to handle when infinity boundary conditions exist as for the problem in this article. Fortunately there are some ways to handle this deficiency, one of which has been used in this article. Homotopy analysis method (HAM) on the other hand is independent of boundary and initial conditions and no matter how the conditions are, it can give a fairy good result. However the process of solution is rather more complex than the other two methods and one should fully understand the introduced parameters and their influences on the equations to achieve a paper answer. These methods can be applied to most of the nonlinear equations in fluid flow. As in this work, the above mentioned methods were applied to a complicated nonlinear system of differential equations.

References

[1] A.H. Nayfeh, Introduction to perturbation techniques, Wiley, 1979.

[2] R.H. Rand, D. Armbruster, Perturbation Methods, Bifurcation theory and Computer Algebra (Applied mathematical Science), Springer, 1987.

[3] A.M. Wazwaz, Adomian decomposition method for a reliable treatment of the Bratu-type equations, Applied Mathematics and Computation, 166 (2005) 652-663.

[4] A.M. Wazwaz, A. Gorgius, Exact Solutions for heat-like and wave-like equations with variable coefficients, Applied Mathematics and Computation, 149 (2004) 15-29.

[5] A.M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Applied Mathematics and Computation, 111 (2000) 53-69.

[6] D. Lesnic, Blow-up solutions obtained using the decomposition method, Chaos, Solitons & Fractals 28 (2006) 776.

[7] S. Saha Ray, A numerical solution of the coupled sine-Gordon equation using the modified decomposition method, Applied Mathematics and Computation, 175 (2006) 1046-1054.

[8] S.J. Liao, Homotopy analysis method: a new analytical technique for nonlinear problems, Communication of Nonlinear Science and Numerical Simulation, 2 (2) (1997) 95-100.

[9] M. Ayub, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, International Journal of Engineering Science, 41 (18) (2003) 2091-2103.

[10] S.J. Liao, On the homotopy analysis method for nonlinear problems, Applied Mathematics and Computation, 147 (2) (2004) 499-513.

[11] S. Liao, Finding multiple solutions of nonlinear problems by means of the homotopy analysis method, Journal of Hydrodynamics, Ser. B, 18 (3) (2006) 54-56.

[12] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, International Journal of Non-Linear Mechanics, 35 (1) (2000) 37-43.

[13] J.H. He, Some asymptotic methods for strongly nonlinear equations, International Journal of Modern Physics B, 20 (10) (2006) 1141-1199.

[14] J. H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons & Fractals, 26 (3) (2005) 695-700.

[15] A. Sadighi, D. D. Ganji, Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods, International Journal of Nonlinear Science and Numerical Simulation, 8 (3) (2007) 435-443.

[16] P.D. Ariel, T. Hayat, S. Asghar, Homotopy perturbation method and axisymmetric flow over a stretching sheet, International Journal of Nonlinear Science and Numerical Simulation, 7 (2006) 399-406.

[17] D. D. Ganji, A. Sadighi, Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations, International Journal of Nonlinear Science and Numerical Simulation, 7 (4) (2006) 411-

[18] M. Rafei, D.D. Ganji, Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method, International Journal of Nonlinear Science and Numerical Simulation, 7 (3) (2006) 321-328.

[19] A. Belendez, Y. HernJndez, T. Belendez, E. Fernandez, M.L. Alvarez, C. Neipp, Application of He's homotopy perturbation method to the Duffing-harmonic oscillator, International Journal of Nonlinear Science and Numerical Simulation, 8 (1) (2007) 79-88.

[20] M. Dehghan, F. Shakeri, Solution of a partial differential equations subject to temperature overspecification by He's homotopy perturbation method, Physica Scripta, 75 (2007) 778-787.

[21] D.D. Ganji, A. Sadighi, Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations, Journal of Computational and Applied Mathematics, 207 (2007) 24-34.

[22] A. Sadighi, D.D. Ganji, Exact solutions of Laplace equation by homotopy-perturbation and Adomian decomposition methods, Physics Letters A, 367 (1-2) (2007) 83-87.

[23] M. Dehghan, F. Shakeri, Solution of an Integro-differential equation arising in oscillating magnetic field using He's homotopy perturbation method, Progress in Electromagnetics Research, 78 (2008) 361-376.

[24] F. Shakeri, M. Dehghan, Inverse problem of diffusion equation by He's homotopy perturbation method, Physica Scripta, 75 (2007) 551-556.

[25] J.H. He, Variational iteration method- a kind of non-linear analytical technique: some examples, International Journal of Non-Linear Mechanics, 34 (4) (1999) 699-708.

[26] J.H. He, Variational iteration method- some recent results and new interpretations, Journal of Computational and Applied Mathematics, 207 (1) (2007) 3-17.

[27] A. Sadighi, D.D. Ganji, Exact solutions of nonlinear diffusion equations by variational iteration method, Computers & Mathematics with Applications, 54 (2007) 1112-1121.

[28] L. Xu, Variational approach to solitons of nonlinear dispersive K(m,n) equations, Chaos, Solitons & Fractals, 37 (1) (2008) 137-143.

[29] J.H. He, X.H. Wu, Construction of solitary solution and compaction-like solution by variational iteration method, Chaos, Solitons & Fractals, 29 (2006) 108-113.

[30] M. Tatari, M. Dehghan, Solution of problems in calculus of variations via He's variational iteration method, Physics Letters A, 362 (200) 401-406.

[31] F. Yusufoglu, Variational iteration method for construction of some compact and noncompact structures of Klein's Gordon equations, International Journal of Nonlinear Science and Numerical Simulation, 8 (2) (200) 152-158.

[32] N.H. Sweilarn, M. Khader, Numerical studies for a multi-order fractional differential equation, Chaos, Solitons & Fractals, 32 (2007) 145-149.
[33] H. Khaleghi, D.D. Ganji, A. Sadighi, Application of variational iteration and homotopy-perturbation methods to nonlinear heat transfer equations with variable coefficients, Numerical Heat Transfer, Part A 52 (2007) 25-42.
[34] M. Dehghan, F. Shakeri, Application of He' variational iteration method for solving the Cauchy reaction diffusion problem, Journal of Computational and Applied Mathematics, 214 (2008) 435-446.
[35] F. Shakeri, M. Dehghan, Numerical solution of the Klein Gordon equation via He's variational iteration method, Nonlinear Dynamics, 51 (2008) 89-97.
[36] M. Tatari, M. Dehghan, On the convergence of He's variational iteration method, Journal of Computational and Applied Mathematics, 207 (2007) 121.
[37] J.H. He, Variational iteration method for autonomous ordinary differential systems, Applied Mathematics and Computation, 114 (2000) 115-123.
[38] S. Momani, S. Abuasad, Application of He's variational-iteration method to Helmholtz equation, Chaos, Solitons & Fractals, 27 (5) (2006) 1119-1123.
[39] S. Abbasbandy, An approximation solution of a nonlinear equation with Riemann Liouville fractional derivatives by He's variational iteration method, Journal of Computational and Applied Mathematics, 207 (1) (2007) 53-58.
[40] N. Bildik, A. Konuarlp, The use of variatial iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, International Journal of Nonlinear Science and Numerical Simulation, 7 (2006) 65-70.
[41] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation, 151 (2004) 287-292.
[42] D.D Ganji, A. Rajabi, Assessment of homotopy-perturbation and perturbation methods in heat radiation equations, International Communications in Heat and Mass Transfer, 33 (3) (2006) 391-400.
[43] D.D. Ganji, The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Physics Letters A, 355 (4-5) (2006) 337.
[44] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[45] S.J. Liao, Auniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate, Journal of Fluid Mechanics, 385 (1999) 101-128
[46] S.J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems, International Journal of Non-Linear Mechanics, 34 (1999) 795-778.
[47] S.J. Liao, On the analytic solution of magnetohyrodynamic flows of non-Newtonian fluids over a stretching sheet, Journal of Fluid Mechanics, 488 (2003) 189-212.
[48] S.J Liao, Beyound Perturbation: Introduction to the Homotopy Anlysis Method, Chapman and Hall/CRC Press, Boca Raton, 2003.
[49] S.J. Liao, On the homotopy analysis method for nonlinear problems, Journal of Applied Mathematics and Computation, 147 (2004) 499-513.
[50] S.J. Liao, Comparison between the homotopy analysis method and homotopy perturbation method, Journal of Applied Mathematics and Computation, 169 (2005) 1186-1194.
[51] F.M. White, Viscous Fluid Flow, McGrow-Hill, 1991.
[52] J.H. He, Non-Perturbative Methods for Strongly Nonlinear Problems, Dissertation.de-Verlag im Internet GmbH, Berlin, 2006.
[53] M. Inokuti, et al., General use of the Lagrange multiplier in non-linear mathematical physics, in: S. Nemat-Nasser (Ed.), Variational Method in the Mechanics of Solids, Pergamon, Oxford (1978) 156-162.
[54] J.H. He, Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamic, International Journal of Turbo and Jet Engines, 14 (1) (1997) 23-28.
[55] B.A. Finlayson, The Method of Weighted Residuals and Variational Principles, Academic Press, New York, 1972.