New Trends in Mathematical Sciences
http://www.ntmsci.com

Parallel Surfaces of Spacelike Ruled Weingarten Surfaces in Minkowski 3-space

Yasin Unluturk[1] *and Cumali Ekici*[2]

[1]Department of Mathematics, Kirklareli University, 39060, Kirklareli, Turkey
[2]Department of Mathematics and Computer Sciences, Eskisehir Osmangazi University, 26480, Eskisehir, Turkey

Received: 2 June 2012, Revised: 23 September 2012, Accepted: 2 November 2012
Published online: 2 January 2013

Abstract: In this work, it is shown that parallel surfaces of spacelike ruled surfaces which are developable ones, are spacelike ruled Weingarten surfaces. It is also shown that parallel surfaces of non-developable ruled Weingarten surfaces are again Weingarten surfaces. Finally, some properties of these kind parallel surfaces are obtained in Minkowski 3-space.

Keywords: Spacelike Weingarten surface, parallel surface, spacelike ruled Weingarten surface, curvatures.

1 Introduction

A Weingarten (or W-) surface is a surface on which there exists a relationship between the principal curvatures. Let f and g be smooth functions on a surface M in Minkowski 3-space. The Jacobi function $\Phi(f,g)$ formed with f,g is defined by $\Phi(f,g) = \det\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ where $f_u = \dfrac{\partial f}{\partial u}$ and $f_v = \dfrac{\partial f}{\partial v}$. In particular, a surface satisfying the Jacobi condition $\Phi(K,H) = 0$ with respect to the Gaussian curvature K and the mean curvature H is called a Weingarten surface or W-surface. All developable surfaces $(K = 0)$ and minimal surfaces $(H = 0)$ are obviously Weingarten surfaces. Some geometers have studied Weingarten surfaces and obtained many interesting results in both Euclidean and Minkowskian spaces [1,2,3,4,5,6,7,11,12,13,19,20].

A surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. Hence there are infinite number of surfaces because we choose the constant distance along the normal, arbitrarily. From the definition, it follows that a parallel surface can be regarded as the locus of point, which are on the normals to M at a non-zero constant distance r from M.

In this paper, we study on parallel surfaces of spacelike surfaces which become both ruled and Weingarten surfaces. We show that parallel surfaces of ruled Weingarten surfaces are again Weingarten surface in Minkowski 3-space. Also, some properties of these kind parallel surfaces are given in Minkowski 3-space.

2 Preliminaries

Let $\mathbb{E}^3_1$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$< d\mathbf{x}, d\mathbf{x} > = dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in R^3. An arbitrary vector $\mathbf{x}$ of $\mathbb{E}^3_1$ is said to be spacelike if $< \mathbf{x}, \mathbf{x} > > 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $< \mathbf{x}, \mathbf{x} > < 0$ and lightlike or null if $< \mathbf{x}, \mathbf{x} > = 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or light-like vector

* Corresponding author e-mail: yasinunluturk@kirklareli.edu.tr

in $\mathbb{E}_1^3$ is said to be causal. For $\mathbf{x} \in \mathbb{E}_1^3$, the norm is defined by $\|\mathbf{x}\| = \sqrt{|<\mathbf{x},\mathbf{x}>|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $<\mathbf{x},\mathbf{x}>=1$ and a timelike unit vector if $<\mathbf{x},\mathbf{x}>=-1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [15]. For any two vectors $\mathbf{x} = (x_1,x_2,x_3)$ and $\mathbf{y} = (y_1,y_2,y_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $<\mathbf{x},\mathbf{y}>= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $\mathbf{x}\times\mathbf{y} = ((x_2y_3 - x_3y_2),(x_3y_1 - x_1y_3),-(x_1y_2 - x_2y_1))$ [14].

A surface in the Minkowski 3-space $\mathbb{E}_1^3$ is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric. This is equivalent to saying that the normal vector on the spacelike surface is a timelike vector [16].

A (differentiable) one-parameter family of (straight) lines $\{\alpha(u),X(u)\}$ is a correspondence that assigns each $u \in I$ to a point $\alpha(u) \in \mathbb{E}_1^3$ and a vector $X(u) \in \mathbb{E}_1^3$, $X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ depend differentiable on u. For each $u \in I$, the line L_u which passes through $\alpha(u)$ and parallel to $X(u)$ is called the *line of the family* at u.

Given a one-parameter family of lines $\{\alpha(u),X(u)\}$, the parametrized surface

$$\varphi(u,v) = \alpha(u) + vX(u), \quad u \in I, \quad v \in \mathbb{R} \tag{2.1}$$

is called the ruled surface generated by the family $\{\alpha(u),X(u)\}$. The lines L_u are called the *rulings* and the curve $\alpha(u)$ is called a *directrix* of the surface φ. The normal vector of surface is denoted by $\overrightarrow{N}$ [16].

Theorem 2.1. Using standard parameters, a ruled surface in Minkowski 3-space is up to Lorentzian motions, uniquely determined by the following quantities:

$$Q = \langle \alpha', X \wedge X' \rangle, \; J = \langle X, X'' \wedge X' \rangle, \; F = \langle \alpha', X \rangle \tag{2.2}$$

each of which is a function of u. Conversely, every choice of these three quantities uniquely determines a ruled surface [13].

Theorem 2.2. The Gauss and mean curvatures of spacelike ruled surface φ in terms of the parameters Q,J,F,D in $\mathbb{E}_1^3$ are obtained as follows:

$$K = \frac{Q^2}{D^4} \quad \text{and} \quad H = \frac{1}{2D^3}[QF - Q^2J - vQ' - v^2J], \tag{2.3}$$

where $D = \sqrt{-\varepsilon Q^2 + \varepsilon v^2}$ [4].

Definition 2.3. A surface is called a Weingarten surface or W-surface in $\mathbb{E}_1^3$ if there is a nontrivial relation $\Phi(K,H) = 0$ or equivalently if the gradients of K and H are linearly dependent. In terms of the partial derivatives with respect to u and v, this is the equation

$$K_uH_v - K_vH_u = 0 \tag{2.4}$$

where K and H are Gaussian and mean curvatures of surface, respectively [4].

Theorem 2.4. The ruled surface φ is a Weingarten surface if and only if the invariants Q, J and F are constants in $\mathbb{E}_1^3$ [4].

Theorem 2.5. Parameter curves are lines of curvature if and only if $F = f = 0$ in $\mathbb{E}_1^3$ [14].

Lemma 2.6. A point $\mathbf{p}$ on a surface M in $\mathbb{E}_1^3$ is an umbilical point if and only if

$$\frac{E}{e} = \frac{F}{f} = \frac{G}{g} \tag{2.5}$$

[10].

Definition 2.7. Let M and M^r be two surfaces in Minkowski 3-space. The function

$$\begin{aligned} f : M &\longrightarrow M^r \\ p &\longrightarrow \ f(p) = p + r\mathbf{N}_p \end{aligned}$$

is called the parallelization function between M and M^r and furthermore M^r is called parallel surface to M in $\mathbb{E}^3_1$ where r is a given positive real number and $\mathbf{N}$ is the unit normal vector field on M [8].

The representation of points on M^r can be obtained by using the representations of points on M in Minkowski 3-space. Let φ be the position vector of a point P on M and φ^r be the position vector of a point $f(P)$ on the parallel surface M^r. Then $f(P)$ is at a constant distance r from P along the normal to the surface M. Therefore the parametrization of M^r is given by

$$\varphi^r(u,v) = \varphi(u,v) + r\mathbf{N}(u,v) \tag{2.6}$$

where r is a constant scalar and $\mathbf{N}$ is the unit normal vector field on M [17].

Theorem 2.8. Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f : M \to M^r$ be the parallelization function. Then for $X \in \chi(M)$,

1. $f_*(X) = X + rS(X)$
2. $S^r(f_*(X)) = S(X)$
3. f preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{k}{1+rk} f_*(X)$$

where S^r is the shape operator on M^r, and k is a principal curvature of M at p in direction X [8].

Theorem 2.9. Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f : M \to M^r$ be the parallellization function. Then f preserves becoming umbilical point on the surface M^r in Minkowski 3-space [18].

Theorem 2.10. Let M be a spacelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. Then we have

$$K^r = \frac{K}{1-2rH-r^2K} \text{ and } H^r = \frac{H+rK}{1-2rH-r^2K} \tag{2.7}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [17].

Corollary 2.11. Let M be a spacelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. Then we have

$$K = \frac{K^r}{1+2rH^r - r^2K^r} \text{ and } H = \frac{H^r - rK^r}{1+2rH^r - r^2K^r} \tag{2.8}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [17].

Theorem 2.12. Let M be a spacelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. Parallel surface of a spacelike developable ruled surface is again a spacelike ruled surface [17].

Theorem 2.13. Let $\varphi(u,v)$ be a spacelike surface in $\mathbb{E}^3_1$ with $F = f = 0$. Then the parallel surface

$$\varphi^r(u,v) = \varphi(u,v) + rN(u,v)$$

is a developable ruled surface while one of the parameters of parallel surface is constant and the other is variable [17].

Corollary 2.14. Let M be a spacelike ruled surface and M^r be a spacelike parallel surface of M in $\mathbb{E}^3_1$. The Gaussian and mean curvatures, respectively, K^r and H^r are as follows:

$$K^r = \frac{Q^2}{D^4 - rQFD + rQ^2JD + rvQ'D + rv^2JD - r^2Q^2} \tag{2.9}$$

$$H^r = \frac{QFD - Q^2JD - vQ'D - v^2JD + 2rQ^2}{2D^4 - 2rQFD + 2rQ^2JD + 2rvQ'D + 2rv^2JD - 2r^2Q^2} \tag{2.10}$$

in terms of the parameters Q, J, F, D [17].

3 Parallel surfaces of spacelike ruled Weingarten surfaces in $\mathbb{E}_1^3$

Let M^r be a parallel surface to a surface M in Minkowski 3-space. If there is a nontrivial relation as

$$\Phi(K^r, H^r) = 0 \tag{3.1}$$

between the Gaussian curvature K^r and the mean curvature H^r of the parallel surface M^r, the parallel surface M^r is said to be Weingarten surface as in analog to the original surface. In other words, if jacobi determinant as a relation between the Gaussian curvature K^r and the mean curvature H^r of the parallel surface M^r vanishes, the following condition for parallel Weingarten surfaces

$$\Phi(K^r, H^r) = \det \begin{pmatrix} K_u^r & K_v^r \\ H_u^r & H_v^r \end{pmatrix} = K_u^r H_v^r - K_v^r H_u^r = 0 \tag{3.2}$$

is obtained.

Theorem 3.1. Let M be a developable spacelike ruled surface in $\mathbb{E}_1^3$, then parallel surface M^r of M is a spacelike parallel ruled Weingarten surface.

Proof. From Theorem 2.11**,** Parallel surface of developable spacelike ruled surface M is again a developable spacelike ruled surface. Therefore, Gaussian curvature of parallel surface K^r vanishes since $K = 0$ for M. It means that the surface is a Weingarten surface.

Theorem 3.2. Let $\varphi(u,v)$ be a spacelike surface in $\mathbb{E}_1^3$ with $F = f = 0$. Then the parallel surface

$$\varphi^r(u,v) = \varphi(u,v) + rN(u,v)$$

is a ruled Weingarten surface while one of the parameters of parallel surface is constant and the other is variable.

Proof. The surface $\varphi^r(u,v)$ is a developable spacelike ruled surface from Theorem 2.12, hence K^r vanishes by putting $K = 0$ in Theorem 2.10. Consequently, the surface is also Weingarten surface.

Theorem 3.3. Let φ^r be a parallel surface of a spacelike ruled surface φ in Minkowski 3-space. If φ is a Weingarten surface if and only if φ^r is a Weingarten surface.

Proof. $(\Rightarrow)$: If φ is a spacelike ruled surface in $\mathbb{E}_1^3$, then we have to show the equation (3.2) by using the equation (2.4).

First, using the expressions of (2.7) in (3.2), we have

$$K_u^r H_v^r - K_v^r H_u^r = \left(\frac{K}{1-2rH-r^2K}\right)_u \left(\frac{H+rK}{1-2rH-r^2K}\right)_v$$

$$-\left(\frac{K}{1-2rH-r^2K}\right)_v \left(\frac{H+rK}{1-2rH-r^2K}\right)_u$$

$$= \Phi\{[K_u\text{-}2rK_uH + 2rKH_u].[H_v + rK_v\text{-}r^2K_vH + r^2KH_v]$$

$$\text{-}[K_v\text{-}2rK_vH\text{+}2rKH_v].[H_u\text{+}rK_u\text{-}r^2K_uH\text{+}r^2KH_u]\} \tag{3.3}$$

where $\Phi = \dfrac{1}{(1-2rH-r^2K)^4}$. If we make computations in (3.3), we get

$$\begin{aligned} K_u^r H_v^r - K_v^r H_u^r = \Phi\{&K_uH_v + rK_uK_v\text{-}r^2K_uK_vH + r^2KK_uH_v \\ &-2rHK_uH_v - 2r^2K_uK_vH + 2r^3K_uK_vH^2 \\ &-2r^3KHK_uH_v + 2rKH_uH_v + 2r^2KK_vH_u \\ &\text{-}2r^3KHK_vH_u + 2r^3K^2H_uH_v + 2rHK_vH_u \\ &+r^2HK_uK_v\text{-}r^2KK_vH_u\text{-}K_vH_u\text{-}2r^3K^2H_uH_v \\ &+2r^2HK_uK_v\text{-}2r^3H^2K_uK_v + 2r^3KHK_vH_u \\ &\text{-}2rKH_uH_v\text{-}r^2KK_uH_v\text{+}2r^3KHK_uH_v\text{-}rK_uK_v\}. \end{aligned} \tag{3.4}$$

After making arrangements in (3.4), the equation becomes as

$$\begin{aligned} K_u^r H_v^r - K_v^r H_u^r = &\frac{1}{(1-2rH-r^2K)^4}\{[K_uH_v - K_vH_u] \\ &-2r[K_uH_v - K_vH_u] - r^2K[K_uH_v - K_vH_u]\}. \end{aligned} \tag{3.5}$$

Later, using (2.4) in (3.5),

$$K_u^r H_v^r - K_v^r H_u^r = 0 \tag{3.6}$$

is found. Parallel surface φ^r is a Weingarten surface since (3.6).

($\Leftarrow$): Conversely, let φ^r be a Weingarten surface which is parallel to a spacelike ruled surface, then it satisfies (3.2). Hence, the equation (2.4) has to be shown. First, using the expressions of (2.8) in (2.4), we have

$$\begin{aligned} K_uH_v - K_vH_u = &\left(\frac{K^r}{1+2rH^r - r^2K^r}\right)_u \left(\frac{H^r - rK^r}{1+2rH^r - r^2K^r}\right)_v \\ &-\left(\frac{K^r}{1+2rH^r - r^2K^r}\right)_v \left(\frac{H^r - rK^r}{1+2rH^r - r^2K^r}\right)_u \end{aligned}$$

$$= \Psi\{[K_u^r + 2rK_u^rH^r - 2rK^rH_u^r].[H_v^r\text{-}rK_v^r\text{-}r^2K_v^rH^r + r^2K^rH_v^r]$$

$$\text{-}[K_v^r\text{+}2rK_v^rH^r\text{-}2rK^rH_v^r].[H_u^r\text{-}rK_u^r\text{-}r^2K_u^rH^r\text{+}r^2K^rH_u^r]\} \tag{3.7}$$

where $\Psi = \dfrac{1}{(1+2rH^r \ -r^2K^r)^4}$. If we make computations in (3.6), we get

$$\begin{aligned} K_uH_v - K_vH_u = \Psi\{ & K_u^rH_v^r - rK_u^rK_v^r - r^2H^rK_u^rK_v^r + r^2K^rK_u^rH_v^r \\ & + 2rH^rK_u^rH_v^r - 2r^2H^rK_u^rK_v^r - 2r^3(H^r)^2K_u^rK_v^r \\ & + 2r^3K^rH^rK_u^rH_v^r - 2rK^rH_u^rH_v^r + 2r^2K^rK_v^rH_u^r \\ & + 2r^3K^rH^rK_v^rH_u^r - 2r^3(K^r)^2H_u^rH_v^r - K_v^rH_u^r \\ & + rK_u^rK_v^r + r^2H^rK_u^rK_v^r - r^2K^rK_v^rH_u^r \\ & - 2rH^rK_v^rH_u^r + 2r^2H^rK_u^rK_v^r + 2r^3(H^r)^2K_u^rK_v^r \\ & - 2r^3K^rH^rK_v^rH_u^r + 2rK^rH_u^rH_v^r - 2r^2K^rK_u^rH_v^r \\ & - 2r^3K^rH^rK_u^rH_v^r + 2r^3(K^r)^2H_u^rH_v^r\}. \end{aligned} \tag{3.8}$$

Making arrangements in (3.8), we get the following equation:

$$\begin{aligned} K_uH_v - K_vH_u = & \frac{1}{(1+2rH^r - r^2K^r)^4}\{(K_u^rH_v^r\text{-}K_v^rH_u^r) + r^2K^r(K_u^rH_v^r\text{-}K_v^rH_u^r) \\ & + 2rH^r(K_u^rH_v^r - K_v^rH_u^r) + 2r^2K^r(K_u^rH_v^r - K_v^rH_u^r)\}. \end{aligned} \tag{3.9}$$

By using (3.2) in (3.9)

$$K_uH_v - K_vH_u = 0 \tag{3.10}$$

is found. Since (3.6), spacelike ruled surface φ is a Weingarten surface.

Corollary 3.4. The surface φ^r which is parallel to spacelike ruled surface φ is a Weingarten surface if and only if the invariants Q, J, F which determine spacelike ruled surface φ are constant.

Proof. $(\Rightarrow)$: Let parallel surface φ^r be a Weingarten surface. Then from Theorem 3.3, spacelike ruled surface φ is also a Weingarten surface. Hence, the invariants Q, J, F are constants from Theorem 2.4.

$(\Leftarrow)$: Let the invariants Q, J, F be constants, then spacelike ruled surface φ is a Weingarten surface from Theorem 2.4. The parallel surface φ^r is also a Weingrten surface from Theorem 3.3.

Corollary 3.5. Let the surfaces φ and φ^r be, respectively, spacelike ruled Weingarten surface and its parallel surface in $\mathbb{E}_1^3$. Then, there is the relation

$$H^r = (r + \frac{H}{K})K^r \tag{3.11}$$

among Gauss K^r and mean H^r curvatures of spacelike parallel Weingarten surface φ^r and Gauss K and mean H curvatures of spacelike ruled Weingarten surface φ.

Proof. We will use the values of the curvatures K^r and H^r given in (2.9) and (2.10). Let

$$A = D^4 - rQFD + rQ^2JD + rvQ'D + rv^2JD - r^2Q^2. \tag{3.12}$$

By using (3.12) in (2.9), we get

$$A = \frac{Q^2}{K^r} \tag{3.13}$$

or

$$A = \frac{QFD - Q^2JD - vQ'D - v^2JD + 2rQ^2}{2H^r} \tag{3.14}$$

is found. From (3.13) and (3.14),

$$H^rQ^2 = [(QF - Q^2J - vQ' - v^2J)D + 2rQ^2]K^r \tag{3.15}$$

is obtained. By using the expressions of Theorem 2.2 in (3.15), we have

$$H^r = \frac{2D^4H + 2rQ^2}{2Q^2}K^r. \tag{3.16}$$

By using the expressions of Theorem 2.2 in (3.16), we get

$$H^r = (r + \frac{H}{K})K^r.$$

Lemma 3.6. Let φ be a non-developable spacelike ruled surface and φ^r be parallel surface of φ in $\mathbb{E}_1^3$. Then there is no umbilical point on spacelike parallel Weingarten surface φ^r.

Proof. Let spacelike ruled Weingarten surface φ be given as the following parameterization:

$$\varphi(u,v) = \alpha(u) + vX(u),\ \langle\alpha',\alpha'\rangle = 1,\ \langle X,X\rangle = 1, \langle X',X'\rangle = \varepsilon, \tag{3.17}$$

where $\varepsilon = \pm 1$ in $\mathbb{E}_1^3$. From Theorem 2.4, the invariants Q, J, F have to be constant for ruled surface to become Weingarten surface. If there is a umbilical point on spacelike ruled Weingarten surface, from Lemma 2.6, it has to be

$$\frac{E}{e} = \frac{F}{f} = \frac{G}{g}. \tag{3.18}$$

Coefficients of first fundamental I for the surface φ are as follows:

$$E = \langle\varphi_u,\varphi_u\rangle = 1 + \varepsilon v^2,\ F = \langle\varphi_u,\varphi_v\rangle = \langle\alpha',X'\rangle,\ G = \langle\varphi_v,\varphi_v\rangle = 1. \tag{3.19}$$

Thereby normal vector of the surface φ is $N = \alpha' \wedge X + vX' \wedge X$. Coefficients of second fundamental II for the surface φ are obtained as

$$e = \langle\varphi_{uu},N\rangle = \langle\alpha'',\alpha' \wedge X\rangle + v\langle\alpha'',X' \wedge X\rangle + v\langle X'',\alpha' \wedge X\rangle + v^2\langle X'',X' \wedge X\rangle,$$

$$f = \langle\varphi_{uv},N\rangle = \langle\alpha',X \wedge X'\rangle \tag{3.20}$$

and

$$g = \langle\varphi_{vv},N\rangle = 0. \tag{3.21}$$

By using (3.19), (3.20) and (3.21) in (2.5), we have

$$Fg - Gf = -\langle X,X\rangle\langle\alpha',X \wedge X'\rangle. \tag{3.22}$$

The equation (3.22) means that there is no umbilical point on a non-developable spacelike ruled surface φ. Hence there is also no umbilical point on parallel surface φ^r of φ from Theorem 2.9.

Example 3.7. Let's take helicoidal surface given as the following parameterization:

$$\varphi(u,v) = (v\sinh u, v\cosh u, u). \tag{3.23}$$

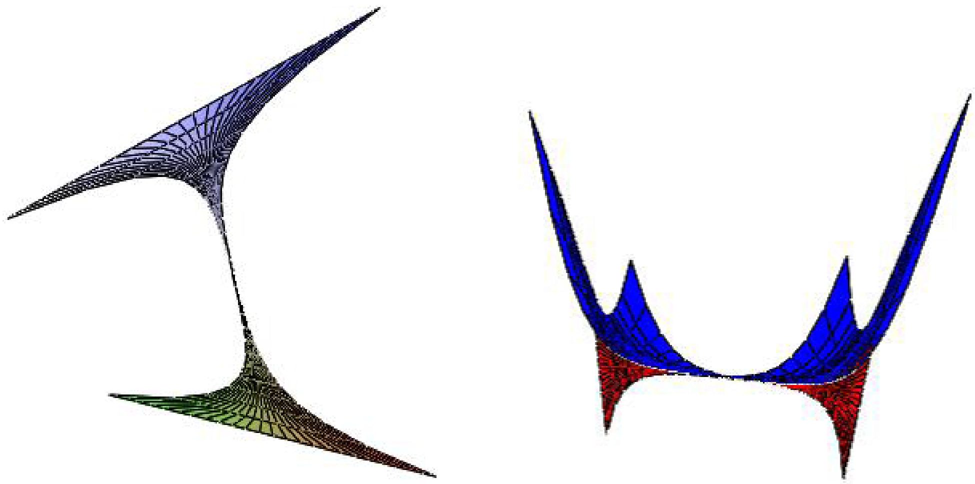
This surface is a spacelike surface for the interval $-1 < v < 1$. If we compute the values of Q, J and F in Theorem 2.2 for that surface, then

$$Q = -1,\ J = 0 \text{ and } F = 0 \tag{3.24}$$

are obtained. The invariants Q, J and F in (3.23) are constant therefore, from Corollary 3.5, parallel surface φ^r is a Weingarten surface. The parametric equation of φ^r which is parallel to spacelike ruled Weingarten surface φ given in (3.23) is obtained as follows

$$\varphi^r(u,v) = (v\sinh u + \frac{r\cosh u}{\sqrt{1-v^2}}, v\cosh u + \frac{r\sinh u}{\sqrt{1-v^2}}, u + \frac{vr}{\sqrt{1-v^2}}). \tag{3.25}$$

The figures 1. (a) and (b) show, respectively, spacelike ruled Weingarten surface φ given in (3.23) and its parallel together with the original surface. The blue surface in Figure 1 (b) show parallel surface, and the red one is for the original surface.



(a) **(b)**

Fig. 1: Spacelike ruled Weingarten surface φ given in (3.23) and its parallel together with the original surface.

References

[1] Beltrami, E., *Risoluzione di un problema relativo alla teoria delle superficie gobbe*, Ann. Mat. Pura Appl., 7, 139-150., 1865.

[2] Brunt, B., *Weingarten surfaces design and application of curves and surfaces*, Fisher, R., (Ed.), Mathematics of surfaces V, Oxford Univ. Press., 1994.

[3] Brunt, B. and Grant, K., *Potential applications of Weingarten surfaces in CAGD. I: Weingarten surfaces and surface shape investigation*, Comput. Aided Geom. Des., 13, 569-582., 1996.

[4] Dillen, F. and Kühnel, W., *Ruled Weingarten surfaces in Minkowski 3-space*, Manuscripta Math., 98, 307–320., 1999.

[5] Dillen, F. and Sodsiri, W., *Ruled surfaces of Weingarten type in Minkowski 3-space*, J. Geom., 83, 10-21., 2005.

[6] Dillen, F. and Sodsiri, W., *Ruled surfaces of Weingarten type in Minkowski 3-space-II*, J. Geom., 84, 37-44., 2005.

[7] Dini, U., *Sulle superficie gobbe nelle quali uno dei due raggidi curvatura principale e una funzione dellaltro*, Ann. Mat. Pura Appl., 7, 205-210., 1865.

[8] Görgülü, A. and Çöken, C., *The Dupin indicatrix for parallel pseudo-Euclidean hypersurfaces in pseudo-Euclidean space in semi-Euclidean space* R^n_1. Journ. Inst. Math. and Comp. Sci. (Math Series), 7(3), 221-225., 1994.

[9] Gray, A., *Modern differential geometry of curves and surfaces*, CRC Press, Inc., 1993.

[10] Hou, Z. H. and Ji, F., *Helicoidal surfaces with* $H^2 = K$ *in Minkowski 3-space*, J. Math. Anal. Appl. 325, 101–113., 2007.

[11] Kim, M. H. and Yoon, D. W., *Weingarten quadric surfaces in a Euclidean 3-space*, Turk. J. Math. 34, 1–7., 2010.

[12] Koch, R., *Die Weingarten-regelflächen*, J. Geom. 47, 77-85., 1993.

[13] Kühnel, W., *Ruled W-surfaces*, Arch. Math. (Basel), 62, 475–480., 1994.

[14] Lopez, R., Differential geometry of curves and surfaces in Lorentz-Minkowski space, Mini-Course taught at the Instituto de Matematica e Estatistica (IME-USP), University of Sao Paulo, Brasil, 2008.

[15] O'Neill, B., Semi Riemannian geometry with applications to relativity, Academic Press, Inc. New York, 1983.

[16] Turgut, A. and Hacısalihoğlu, H. H., *Spacelike ruled surfaces in the Minkowski 3-space*, Commun. Fac. Sci. Univ. Ank. Series A1, v. 46, pp. 83-91, 1997.

[17] Unluturk, Y., and Ekici, C., *On spacelike parallel ruled surfaces in Minkowski 3-space,* (Preprint).

[18] Unluturk, Y., Ekici, C., and Özüsağlam, E., *On parallel surfaces in Minkowski 3-space,* (Preprint).

[19] Weingarten, J., *Über eine klasse auf einander abwickelbarer flächen*, J. Reine Angew. Math. 59, 382-393, 1861.

[20] Weingarten, J., *Über eine flächen, derer normalen eine gegebene fläche-berühren*, Journal für die Reine und Angewandte Mathematik, 62, 61-63, 1863.

[21] D. W. Yoon, *Some properties of parallel surfaces in Euclidean 3-spaces*, Honam Mathematical J. 30, No. 4, pp, 637-644, 2008.